Recall: $Y_{ij} \overset{iid}{\sim} N(\mu_i, \sigma^2) \;\rightarrow\; E\left[\sum_{j=1}^{n}\left(Y_{ij}-\bar{Y}_{i\bullet}\right)^2\right]=(n-1)\sigma^2$

	Source		d.f.	SS	MS	F
(Between Groups)	Treatment	(A)	*t-1*	$\sum_{i=1}^{t}\sum_{j=1}^{n_i}\left(\bar{Y}_{i\bullet}-\bar{Y}_{\bullet\bullet}\right)^2$	*SST / (t-1)*	*MST / MSE*
(Within Groups)	Error	(W)	*N-t*	$\sum_{i=1}^{t}\sum_{j=1}^{n_i}\left(Y_{ij}-\bar{Y}_{i\bullet}\right)^2$	*SSE / (N-t)*	
	Total		*N-1*	$\sum_{i=1}^{t}\sum_{j=1}^{n_i}\left(Y_{ij}-\bar{Y}_{\bullet\bullet}\right)^2$		

Random Effects Model: $Y_{ij}=\mu+a_i+\varepsilon_{ij} \qquad a_i \sim N(0,\sigma_a^2), \quad \varepsilon_{ij}\sim N(0,\sigma_e^2)$

$$\bar{Y}_{i\bullet}=\mu+a_i+\bar{\varepsilon}_{i\bullet}$$
$$\bar{Y}_{\bullet\bullet}=\mu+\bar{a}_{\bullet}+\bar{\varepsilon}_{\bullet\bullet}$$

$$SSW=\sum_{i=1}^{t}\sum_{j=1}^{r}\left(Y_{ij}-\bar{Y}_{i\bullet}\right)^2 = \sum_{i=1}^{t}\sum_{j=1}^{r}\left[(\mu+a_i+\varepsilon_{ij})-(\mu+a_i+\bar{\varepsilon}_{i\bullet})\right]^2$$
$$= \sum_{i=1}^{t}\sum_{j=1}^{r}\left[\varepsilon_{ij}-\bar{\varepsilon}_{i\bullet}\right]^2$$

$$SSA=\sum_{i=1}^{t}\sum_{j=1}^{r}\left(\bar{Y}_{i\bullet}-\bar{Y}_{\bullet\bullet}\right)^2 = \sum_{i=1}^{t}\sum_{j=1}^{r}\left[(\mu+a_i+\bar{\varepsilon}_{i\bullet})-(\mu+\bar{a}_{\bullet}+\bar{\varepsilon}_{\bullet\bullet})\right]^2$$
$$= \sum_{i=1}^{t}\sum_{j=1}^{r}\left[(a_i-\bar{a}_{\bullet}) \;+\; (\bar{\varepsilon}_{i\bullet}-\bar{\varepsilon}_{\bullet\bullet})\right]^2$$
$$= \cdots$$

Recall: $\sum_{j=1}^{n} z_i^2 \sim \chi_n^2 \quad \rightarrow \quad \sum_{j=1}^{n}\left(\frac{Y_i-\mu}{\sigma}\right)^2 \sim \chi_n^2$

$$\sum_{j=1}^{n}\left(\frac{Y_i-\bar{Y}}{\sigma}\right)^2 \sim \chi_{n-1}^2$$

$$(n-1)\frac{\sum_{j=1}^{n}(Y_i-\bar{Y})^2/(n-1)}{\sigma\;\;2} \sim \chi_{n-1}^2$$

$100(1-\alpha)\%$ CI for σ_e^2

$$\left(\frac{(N-t)MSE}{\chi^2_{\frac{\alpha}{2},N-t}} \,,\, \frac{(N-t)MSE}{\chi^2_{1-\frac{\alpha}{2},N-t}} \right)$$

$100(1-2\alpha)\%$ CI for σ_a^2

$$\left(\frac{SSA\left[1-F_{\frac{\alpha}{2},t-1,N-t}\dfrac{MSW}{MSA}\right]}{r\chi^2_{\frac{\alpha}{2},t-1}} \,,\, \frac{SSA\left[1-F_{1-\frac{\alpha}{2},t-1,N-t}\dfrac{MSW}{MSA}\right]}{r\chi^2_{1-\frac{\alpha}{2},t-1}} \right)$$

$F.E.M.$: $Y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \qquad \sum_i \alpha_i = 0, \qquad \varepsilon_{ij} \sim N(0,\sigma_e^2)$

$R.E.M.$: $Y_{ij} = \mu + a_i + \varepsilon_{ij} \qquad a_i \sim N(0,\sigma_a^2), \quad \varepsilon_{ij} \sim N(0,\sigma_e^2)$

$$Power = P\left(\text{Reject } \mathrm{H}_0 \mid \mathrm{H}_1\right) = P\left(F > \frac{1}{\lambda^2} F_{\alpha,\nu_1,\nu_2} \mid \sigma_a^2 > 0 \right)$$

$$\lambda^2 = 1 + r\frac{\sigma_a^2}{\sigma_e^2} = 1 + r\left(\frac{\sigma_y^2 - \sigma_e^2}{\sigma_e^2}\right) = 1 + r\left(\frac{\sigma_y^2}{\sigma_e^2} - 1\right)$$

$$= \qquad 1 + r\left(\left(1+P_e\right)^2 - 1\right)$$

$$Y_{ijk} = \mu + \alpha_i + e_{ij} + d_{ijk} \qquad \sum_{i=1}^{t} \alpha_i = 0, \quad e_{ij} \sim NID(0, \sigma_e^2), \quad d_{ijk} \sim NID(0, \sigma_d^2)$$

$i = 1, \ldots, t$ (Trt groups) (Methods)

$j = 1, \ldots, r$ (EUs in i^{th} Trt) (Batches or plots)

$k = 1, \ldots, n$ (SUs in j^{th} EU) (Subsamples)

$$\left(Y_{ijk} - \bar{Y}_{\bullet\bullet\bullet}\right) = \left(\bar{Y}_{i\bullet\bullet} - \bar{Y}_{\bullet\bullet\bullet}\right) + \left(\bar{Y}_{ij\bullet} - \bar{Y}_{i\bullet\bullet}\right) + \left(Y_{ijk} - \bar{Y}_{ij\bullet}\right)$$

	Source	d.f.	SS	MS	E(MS)
(Methods)	**Trt**		*SST*	*MST*	
(Batches)	**"Error"***		*SSE*	*MSE*	
	Sampling**		*SSS*	*MSS*	σ_d^2
	Total	*trn-1*	*SS(Tot)*		

* Listed as ***batch*** [or ***batch(method)***] in SAS, but called "***Error***" in the text.
** Listed as ***Error*** in SAS (unless a fully specified model is used).

Allocation of Sampling Units vs. Experimental Units

$$V_i^* = Var(\bar{Y}_{i\bullet\bullet}) = \frac{\sigma_e^2}{r^*} + \frac{\sigma_d^2}{r^* n^*} \quad \Rightarrow \quad r^* = \frac{\sigma_e^2}{V_i^*} + \frac{\sigma_d^2}{V_i^* n^*}$$

$$C_i^* = Cost_i = C_{EU} \bullet r^* + C_{SU} \bullet r^* n^*$$

Goal: Find n^* in order to minimize C_i^* while holding V_i^* constant.

Source	SS
Model	187
Error	8
C. Total	195

Source	Type I SS
X1	144
X2	22
X3	21

Terms in Model	SSE	Decrease
none	195	
		144
X1	51	
		22
X1,X2	29	
		21
X1,X2,X3	8	

Source	SS
Model	187
Error	8
C. Total	195

Adding X1 to the model that already has X2 and X3 reduces the SSE by 10.

Source	Type III SS
X1	10
X2	37
X3	21

Adding X2 to the model that already has X1 and X3 reduces the SSE by 37.

Adding X3 to the model that already has X1 and X2 reduces the SSE by 21.

Terms in Model	SSE
X1,X2,X3	8
X2,X3	18
X1,X3	45
X1,X2	29

A difference of 37.