Hypothesis Testing

- Null Hypothesis (H_o) vs. Alternative Hypothesis (H_A)
- α = P(Type I Error) → the <u>level of significance</u> (*l.o.s.*)
- β = P(Type II Error)
- *power* = $1-\beta$

p-value

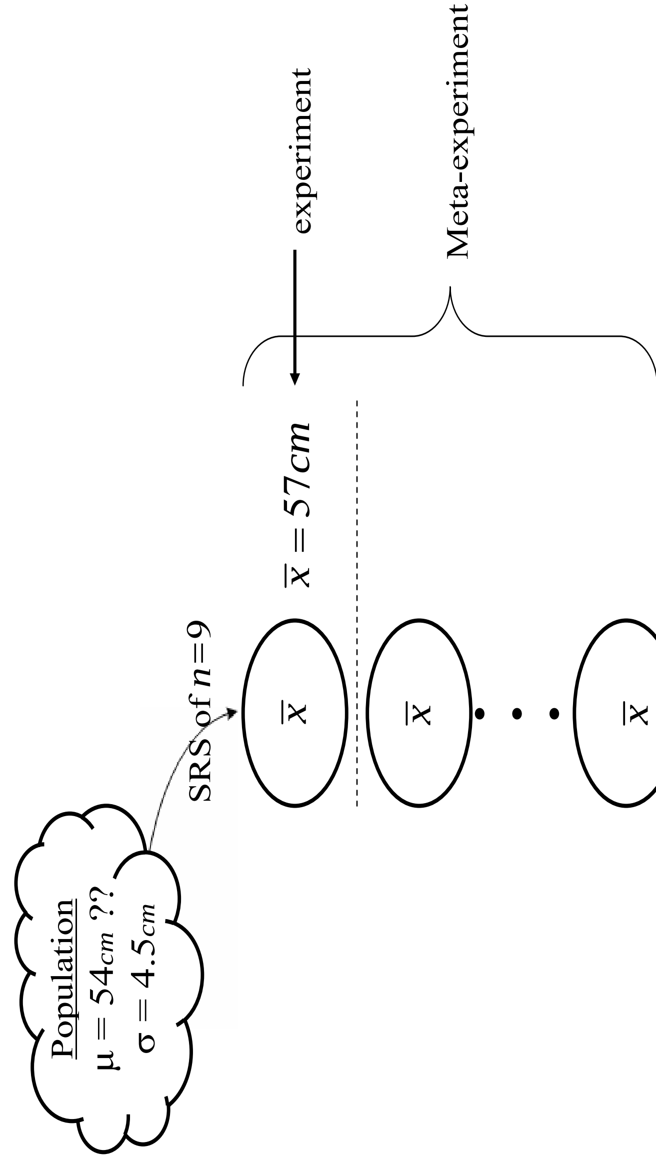
- Measures the strength of the sample evidence against H_o
- <u>Definition</u>:

  The probability, computed assuming that H_o is true, of a sample result ($\bar{X}$) as extreme or more extreme than the one from our sample.

Rule of Thumb for the significance of p-values

- The p-value is the smallest *l.o.s.* α at which we can reject H_o
- If the p-value is less than .05, then our results are <u>statistically significant</u> at the .05 level

Let X be a R.V. denoting the length of a fish in *cm*.



Does $\bar{x} = 57$ give strong evidence that $\mu>54$ *cm*?

How likely are we to get $\bar{x} = 57$ if $\mu=54$ *cm*?

H_o: μ=54 *cm* $\sigma_{\bar{X}} = \frac{4.5\ cm}{\sqrt{9}} = 1.5\ cm$

H_a: μ>54 ($\mu_a = 58$) α = .05

<u>4 Steps for finding the Power in a test of hypotheses</u>

1) Write the RR for H_o in terms of z-scores:

2) Write the RR for H_o in terms of $\bar{X}$:

3) Find the probability of a Type II error <u>if μ=58</u>

4) Power = 1 – P(Type II Error):

H_o: μ = 40 mpg
H_A: μ < 40

Population Standard Deviation: σ = 6 mpg

Significance Level: α = .01

Sample Results: A SRS of $n = 16$ gives $\bar{X} = 36.7$

1) Write the rejection rule (RR) for H_o in terms of z-scores.

2) Write the rejection rule (RR) for H_o in terms of $\bar{X}$.

3) Find the probability of a Type II error if μ=38 $\left[i.e.,\ \beta(38)\right]$

a) Find the sample z-score (z_s).

b) State a conclusion for the test at the α = .01 level.

c) Find the p-value.

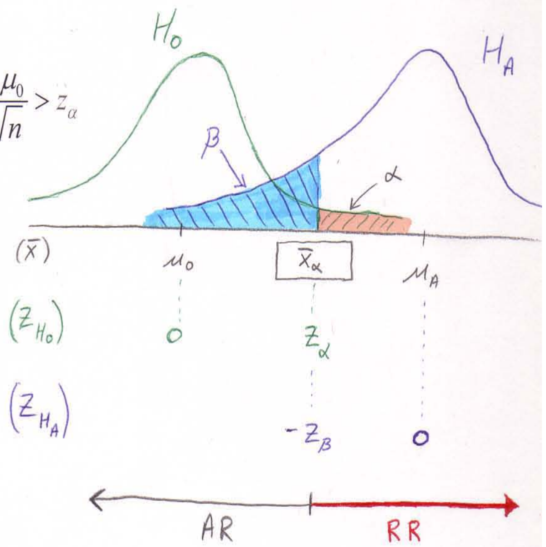
$H_0 : \mu = \mu_0$

$H_A : \mu > \mu_0 \;\; (= \mu_A)$

$RR \text{ in terms of } Z: \;\; Z_s > z_\alpha \;\; \Leftrightarrow \;\; \dfrac{\bar{X} - \mu_0}{\sigma / \sqrt{n}} > z_\alpha$

$RR \text{ in terms of } \bar{X}: \;\; \bar{X} > \underbrace{z_\alpha \dfrac{\sigma}{\sqrt{n}} + \mu_0}_{\bar{X}_\alpha}$

$\beta = P(\text{Accept } H_0 \quad | \; H_A \text{ is true})$



Probability Distributions for Random Variables (RVs)

- Probability Distribution (for a *discrete* RV) – represented graphically as a *Probability Histogram*
  - Indicates the possible values for a RV
  - Indicates how to assign probabilities for the possible values: $p(x) = P(X = x)$
- Probability Distribution (for a *continuous* RV) – represented as a *Probability Density Curve*
  - Areas under a smooth curve, $f(x)$, indicate probabilities of values in a given range

Expected Value of a Random Variable

- a weighted average of all possible values for *X*, weighted by the probability of each value
  $E(X) = \mu$, the mean for the RV
  - $E(X) = \sum_{x=-\infty}^{\infty} x\, p(x)$ for a *discrete* RV
  - $E(X) = \int_{x=-\infty}^{\infty} x\, f(x)\, dx$ for a *continuous* RV

Example (*discrete* RV):
Y = # heads in two tosses of a fair coin

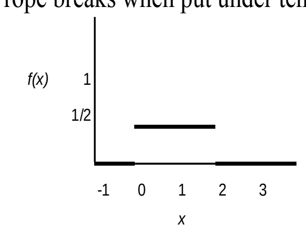
$P(Y = 0) = 1/4$
$P(Y = 1) = 1/2$
$P(Y = 2) = 1/4$ and zero otherwise

$$E(Y) = 0 \bullet P(Y=0) + 1 \bullet P(Y=1) + 2 \bullet P(Y=2)$$

$$= 0 \bullet \frac{1}{4} + 1 \bullet \frac{1}{2} + 2 \bullet \frac{1}{4} = 1$$

Example (*continuous* RV):
X = the position at which a two-meter with length of rope breaks when put under tension (assuming every point is equally likely)

$f(x) = 1/2 \;\; 0 \le x \le 2$, zero otherwise



$$E(X) = \int_0^2 x \left(\frac{1}{2}\right) dx = \frac{1}{4} x^2 \Big|_0^2 = 1 - 0 = 1$$

Properties of Expectation

E(a) $= a$
E(aX) $= a * E(X)$
E(X+a) $= E(X) + a$
E(X+Y) $= E(X) + E(Y)$

$E(XY) = E(X) * E(Y)$ if X & Y are independent

Variance of a Random Variable

- a weighted average of *squared deviations from the mean*, $[x - E(x)]^2$

$$Var(X) = E[(X-\mu)^2] = \sigma^2$$

  - $Var(X) = \sum_{x=-\infty}^{\infty} \left(x - E(x)\right)^2 p(x)$ for a *discrete* RV
  - $Var(X) = \int_{x=-\infty}^{\infty} \left(x - E(x)\right)^2 f(x)\,dx$ for a *continuous* RV

$$Var(X) = E[(X-\mu)^2] \;\Rightarrow\; E(X^2) - [E(X)]^2 \quad \text{for } \textbf{\textit{any}} \text{ RV}$$

Example:

Y = # heads in two tosses of a fair coin

P(Y = 0) = ¼
P(Y = 1) = ½
P(Y = 2) = ¼ and zero otherwise

$$\mu = E(Y) = 1$$

$$Var(Y) = (0-\mu)^2 \bullet P(Y=0) + (1-\mu)^2 \bullet P(Y=1) + (2-\mu)^2 \bullet P(Y=2)$$

$$= (0-1)^2 \bullet \frac{1}{4} \quad + (1-1)^2 \bullet \frac{1}{2} \quad + (2-1)^2 \bullet \frac{1}{4}$$

$$= \frac{1}{2}$$

OR

$$E(Y^2) = 0^2 \bullet P(Y=0) + 1^2 \bullet P(Y=1) + 2^2 \bullet P(Y=2)$$

$$= 0 \bullet \frac{1}{4} \; + \; 1 \bullet \frac{1}{2} \; + \; 4 \bullet \frac{1}{4} \; = \; \frac{3}{2}$$

$$Var(Y) = E(Y^2) - [E(Y)]^2 = \frac{3}{2} - 1^2 = \frac{1}{2}$$

Properties of Variance

Var(X±a) = *Var(X)*

Var(aX) = a^2**Var(X)*

Var(X+Y) = *Var(X) + Var(Y)* if X & Y are independent

Var(X+Y) = *Var(X) + Var(Y) + 2*Cov(X,Y)* always

*Cov(X,Y) = E([X–E(X)] * [Y–E(Y)])*
*= E(XY) – E(X)*E(Y)*

$$Y_1, Y_2, \ldots, Y_n \overset{iid}{\sim} N(\mu, \sigma^2)$$ → a *Simple Random Sample* (SRS) of size *n*

i.i.d. → independent and identically distributed

$$E(\bar{Y}) = E\left(\frac{Y_1 + Y_2 + \ldots + Y_n}{n}\right) = \frac{1}{n} E\left(Y_1 + Y_2 + \ldots + Y_n\right) = \frac{1}{n}(n \bullet \mu) = \mu$$

$$Var(\bar{Y}) = Var\left(\frac{Y_1 + Y_2 + \ldots + Y_n}{n}\right) = \frac{1}{n^2} Var\left(Y_1 + Y_2 + \ldots + Y_n\right) = \frac{1}{n^2}\left(n \bullet Var(Y_i)\right) = \frac{\sigma^2}{n}$$

$$\frac{\bar{Y} - \mu}{\sigma / \sqrt{n}} = Z \sim N(0,1)$$

2-Sample Tests

$$Y_{11}, Y_{12}, \ldots, Y_{1n_1} \overset{iid}{\sim} N(\mu_1, \sigma_1^2)$$

$$H_0 : \mu_1 = \mu_2$$

$$Y_{21}, Y_{22}, \ldots, Y_{2n_2} \overset{iid}{\sim} N(\mu_2, \sigma_2^2)$$

$$E(\bar{Y}_1 - \bar{Y}_2) = E(\bar{Y}_1) - E(\bar{Y}_2) = \mu_1 - \mu_2$$

$$Var(\bar{Y}_1 - \bar{Y}_2) = Var(\bar{Y}_1) + Var(\bar{Y}_2) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$