Genus 5 curves are the intersection of 3 quadrics in $\mathbb{P}^4$

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Notation

X or C	$\cdots$	a curve
S	$\cdots$	a surface
Ω	$\cdots$	used as canonical bundle w/o subscript, or sheaf of holomorphic differentials with subscript
ω_X	$\cdots$	canonical sheaf/bundle
I	$\cdots$	ideal sheaf, resp. (abusively) homogeneous ideal or canonical ideal. Note this is graded.
L	$\cdots$	a line bundle/invertible sheaf on X
φ	$\cdots$	the map to projective space induced by L
$H^0(X,L)$	$\cdots$	the global sections of L on X
K	$\cdots$	or $K_X,$ a canonical divisor on X
$\mathcal{O}$	$\cdots$	the structure sheaf, often abusively used for ring of integers of that sheaf
s_i	$\cdots$	ith global section
φ_i	$\cdots$	ith basis element for group of global sections
$\mathbb{P}^n$	$\cdots$	$\mathrm{Proj}(H^0(X,L))$ for some degree n bundle L
$\bigwedge M$	$\cdots$	for some free module $M,$ the exterior algebra
$\mathrm{Sym}(M)$	$\cdots$	the symmetric algebra on the free module M
$d_{p,q}$	$\cdots$	the boundary map in the (p,q)th part of the Koszul complex
$K_{p,q}(B,V)$	$\cdots$	the (p,q)th Koszul cohomology group of a $\mathrm{Sym}(V)$-module B and vector space V

1. Introduction

This document is intended to demonstrate how to derive the Petri equations for a canonical curve. A classic example of some known equations are the 3 degree 2 equations in $\mathbb{P}^4$ (quadrics) which determine a genus 5 curve by their complete intersection and these notes will start off by writing down those equations. The Petri equations in general are designed to compute the section ring of a curve in an embedding, and the decisive idea which makes it possible to determine in particular what kinds of polynomials generate the homogeneous ideal of an embedded curve is to demonstrate generation of that ideal with Koszul cohomology. That is accomplished here by intersecting the multiplication sequences for the symmetric and tensor algebras over the vector space of global sections with both the Koszul complexes and the exact sequences determined by the normal generation of the canonical bundle. The main results in these notes are Max Noether's theorem that a canonically embedded nonhyperelliptic curve is projectively normal and Theorem 3.1 which is summarized in terms of the title as:

Theorem 1.1. *Every minimal generator for the canonical ideal of nonhyperelliptic, canonical genus* 5 *curve in* $\mathbb{P}^4$ *has degree at most* 2.

First, as promised, this document will actually include the degree 2 homogeneous equations which generate the canonical ideal of the embedded curve. In the next section, many definitions about line bundles are stated, the Euler relations between the structure sheaf and the sheaf of holomorphic differentials is introduced with a couple of important twists and some sheaf cohomology, and the Koszul cohomology is developed. An entire section is devoted to each of the rather elaborate proofs of the main theorems stated above. Finally, some literature review regarding what people have done for other kinds of curves and for which varieties there are known or knowable Petri equations, is the conclusion of the document, by way of alluding to the next projects which will follow this example.

1.1. Motivating example.

Let X be a genus 5 canonical, non-hyperelliptic, smooth, irreducible, projective, complex algebraic curve. The canonical bundle $\Omega = \omega_X$ defines an embedding (closed) $\varphi : X \to \mathbb{P}^4$. Let R denote the canonical ring $R = R(X) = \bigoplus_{d=0}^{\infty} H^0(X, \Omega^{\otimes d})$ of X in $\mathbb{P}^4$. In particular $R \cong \mathbb{C}[x_1, \cdots, x_5]/I$. and the point of this document is to show:
R is genrerared in degree 1, I is generated in degree 2 ('by quadrics'), and $\dim_{\mathbb{C}} I_2 = 3$.

To begin the discussion properly requires at least a working concept of what a syzygy module is. As an R-ideal, I naturally has the structure of an R-module, and in fact is finitely generated. However, more information is needed that simply the generators, say $f_1, \cdots, f_n$ for I. In particular there are nontrivial relations among those generators, which form a set called the (first) syzygies [3, chapter 6], denoted $\mathrm{Syz}(f_1, \cdots, f_n)$. It turns out that $\mathrm{Syz}(f_1, \cdots, f_n)$ is itself an R-module, say with generators $g_1, \cdots, g_m$, and there is an R-module of relations among the g_i, denoted $\mathrm{Syz}(g_1, \cdots, g_m)$, which is the module of (second) syzygies for I. Proceeding in this way one defines a sequence of successive syzygy modules for I which is called a resolution. The properties of that sequence itself which are relevant for this discussion are discussed later in Section 3. With this inductive-like idea of how relations among generators form modules known as syzygies, the syzygies which correspond to the quadrics whose complete intersection is a genus 5 curve in $\mathbb{P}^4$ can be written down explicitly.

Let $\varphi : X \to \mathbb{P}^4$ be the map obtained from global sections of the canonical bundle

$$p \mapsto [s_1(p), \cdots, s_5(p)]$$

and let $x_1, \cdots, x_5 \in X$ be some closed points in general position. Then consider a basis $\varphi_1, \cdots, \varphi_5$ of $H^0(X, \Omega)$ such that $\varphi_i(x_j) \neq 0$ if and only if $i = j$.

In particular by a uniform position theorem in [2, Section 3] and the geometric Riemann-Roch

$$\dim H^0(X, K(-x_1 - \cdots - \hat{x_i} - \cdots - x_g)) = 1,$$

where $\hat{x_i}$ means that point is excluded, and φ_i is taken to be the generator for each i. Therefore as a section of K

$$\begin{cases} \varphi_i(x_i) \neq 0, \\ \varphi_i(x_j) = 0, \quad i \neq j \end{cases}$$

so the φ_i form a basis for $H^0(C, K)$.

The assumption that the points x_i are in general position also means the divisors (φ_i) are supported at $2g-2$ distinct points with pairwise disjoint support. Note that for any relation

$$\sum \lambda_i \varphi_i = 0,$$

evaluating at x_i gives $\lambda_i = 0$

To understand these relations and ultimately to give bases for each graded component of the ideal of X in $\mathbb{P}^4$ some notation and the base point free pencil trick will be introduced next. Consider the maps

$$\psi_n : H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(n)) \to H^0(C, K^n)$$

given by restriction and let $X_1, \cdots, X_g$ be a basis for $H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(1))$ defined by

$$X_i = \psi_1^{-1}(\varphi_i),$$

so that the X_i act like homogeneous coordinates.

Example. [2] Given $P = P(X_1, \cdots, X_g) \in H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(1))$ say that $\overline{P} = \psi_n(P)$ so in paricular one might say

$$\overline{X_1^2 X_3} = \varphi_1^2 \varphi_3.$$

Let $D = x_3 + \cdots + x_g$. Then the general position of the x_i means

$$\dim H^0(X, K(-D)) = 2,$$

where the vector space has a basis φ_1 and φ_2. Since the support of the (φ_i) are pairwise disjoint, the pencil $|K(-D)|$ is basepoint free. Riemann-Roch helps out once again: in the towers

$$H^0(X, K^n) \supset H^0(X, K^n(-D)) \supset \cdots \supset H^0(X, K^n((-n+1)D)),$$

where $n-1 \geqslant s \geqslant 1$, for each s the theorem says

$$h^0(X, K^n(-sD)) = (2n-1)(g-1) - s(g-2)$$

and each vector space in the filtration has codimension $g-2$ in the previous. To actually write Petri's equations for the genus 5 curves, for each s there must be n-canonical forms in $H^0(X, K^n(-sD))$ which are linearly independent modulo $H^0(X, K^n((-s-1)D))$.

Lemma 1.2. [2, Basepoint free pencil trick] *Let C be a smooth curve, let L be an invertible sheaf on C and let $\mathcal{F}$ be a free $\mathcal{O}_C$-module. Suppose s_1 and s_2 are linearly independent sections of L and denote the subspace of $H^0(C, L)$ which they generate V. Then the map*

$$\phi_{2,2} : V \otimes H^0(C, \mathcal{F}) \to H^0(C, \mathcal{F} \otimes L)$$

given by

$$s_1 \otimes t_2 - s_2 \otimes t_1 \mapsto s_1 t_2 - s_2 t_1$$

has kernel

$$\ker \phi_{2,2} \cong H^0(C, \mathcal{F} \otimes L^{-1}(B)),$$

where B is the base locus of the pencil spanned by s_1 and s_2.

The application which is relevant to the Petri equations is

$$\ker \phi_{n,s} \cong H^0(C, K^{n-2}((-s+2)D)).$$

Now the inductive desciption of bases for the $H^0(X, K^n)$ for each n proceeds as follows. The map

$$\phi_{2,1} : H^0(X, K) \otimes H^0(X, K(-D)) \to H^0(X, K^2(-D))$$

is surjective by 1.2 so

$$\varphi_1^2,\ \varphi_1\varphi_2,\ \varphi_2^2,\ \varphi_1\varphi_i,\ \varphi_2\varphi_i,$$

where $3 \leqslant i \leqslant g$, form a basis for $H^0(X, K^2(-D))$. At the top of the tower

$$H^0(X, K^2) \supset H^0(X, K^2(-D)),$$

the $\varphi_3^2, \cdots, \varphi_g^2$ are differentials in $H^0(X, K^2)$ which are linearly independent modulo $H^0(X, K^2(-D))$ and since codim$(H^0(X, K^2(-D))$ in $H^0(X, K^2)) = g - 2$ the basis for $H^0(X, K^2)$ is

$$\begin{array}{c|c|l} \varphi_1^2, \varphi_1\varphi_2, \varphi_2^2 & & \\ \varphi_1\varphi_i & & \text{basis of } H^0(X, K^2(-D)) \\ \varphi_2\varphi_i & & \\ \varphi_3^2, \cdots, \varphi_g^2 & & \quad \text{basis of } H^0(X, K^2). \end{array}$$

However, in writing down all of the differentials in each homogeneous order n, some nontrivial relations begin to arise between them. For example, for all $3 \leqslant i, k \leqslant g$ where $i \neq k$, $\varphi_i\varphi_k \in H^0(X, K^2(-D))$ and in particular vanishes at x_1 and x_2.

Mumford in [16] concisely describes these relations

$$\varphi_i\varphi_j = \sum_{k=3}^{g} \alpha_{ijk}(\varphi_1, \varphi_2)\varphi_k + \nu_{ij}\varphi_1\varphi_2,$$

and among higher orders than explicitly written down here (in $H^0 X, K^3$ in particular)

$$\eta_i - \eta_j = \sum_{k=3}^{g} \alpha'_{ijk}(\varphi_1, \varphi_2)\varphi_k + \nu'_{ij}\varphi_1^2\varphi_2 + \nu''_{ij}\varphi_1\varphi_2^2,$$

where the α are linear, α' are quadratic and ν's are scalars repectively.

In particular the homogeneous degree 2 equations

$$f_{ij} = x_i x_j - \sum_{k=3}^{g} \alpha_{ijk}(x_1, x_2)x_k - \nu_{ij}x_1x_2,$$

and the degree 3 equations

$$g_{ij} = (\mu_i x_1 - \lambda_i x_2)x_i^2 - (\mu_j x_1 - \lambda_j x_2)x_j^2 - \sum_{k=3}^{g} \alpha'_{ijk}(x_1, x_2)x_k - \nu'_{ij}x_1^2x_2 - \nu''_{ij}x_1x_2^2,$$

where the $3 \leqslant i, j \leqslant 5$, and $i \neq j$ are generators of the ideal of X in $\mathbb{P}^4$. In other words the f_{ij} all vanish on X in $\mathbb{P}^{g-1}$ and are exactly the subvariety-defining equations guarenteed by Petri, Enriques, Babbage, et al in [2] and [16]. To be rigorous, these

$$\frac{(g-2)(g-3)}{2}$$

linearly independent elements of I_2 match the dimension of I_2 which [2]'s formulation of Max Noether's theorem guarentees so indeed the f_{ij} form a basis.

The full list of these equations is

$$\begin{array}{cccccc} f_{34}, & f_{35}, & f_{43}, & f_{45}, & f_{53}, & f_{54} \\ g_{34}, & g_{35}, & g_{43}, & g_{45}, & g_{53}, & g_{54} \end{array}$$

These are the equations guarenteed by Max Noether's theorem but there are nontrivial syzygies between these relations.

Lemma 1.3. [16] *Let X be a genus 5 canonical, non-hyperelliptic, smooth, irreducible, complex algebraic curve. There are syzygies*

(1) $f_{ij} = f_{ji}$
(2) $g_{ij} + g_{jk} = g_{ik}$.
(3) $x_k f_{ij} - x_j f_{ik} + \sum_{\substack{l=3 \\ l\neq k}}^{g} \alpha_{ijl} f_{kl} - \sum_{\substack{l=3 \\ l\neq k}}^{g} \alpha_{ikl} f_{jl} = \rho_{ijk} g_{jk}$,

where $3 \leqslant i, j, k \leqslant g$, i, j, k are distinct, and the ρ_{ijk} are scalars symmetric in i, j and k, which generate the components of the homogeneous ideal of X in its canonical embedding $I_{X/\mathbb{P}^{g-1},2}$ and $I_{X/\mathbb{P}^{g-1},3}$ respectively.

Proof. This is a proof of only the second syzygy. The first is trivial and the third requires more discussion.

$$\begin{aligned} g_{ij} + g_{jk} &= (\mu_i x_1 - \lambda_i x_2)x_i^2 - (\mu_j x_1 - \lambda_j x_2)x_j^2 - \textstyle\sum_{k=3}^{g} \alpha'_{ijk}(x_1, x_2)x_k - \nu'_{ij}x_1^2x_2 - \nu''_{ij}x_1x_2^2 \\ &+(\mu_j x_1 - \lambda_j x_2)x_j^2 - (\mu_k x_1 - \lambda_k x_2)x_k^2 - \textstyle\sum_{k=3}^{g} \alpha'_{ijk}(x_1, x_2)x_k - \nu'_{jk}x_1^2x_2 - \nu''_{jk}x_1x_2^2 \\ &= (\mu_i x_1 - \lambda_i x_2)x_i^2 - (\mu_k x_1 - \lambda_k x_2)x_k^2 - \textstyle\sum_{k=3}^{g} \alpha'_{ijk}(x_1, x_2)x_k - \nu'_{ik}x_1^2x_2 - \nu''_{ik}x_1x_2^2 \\ &= g_{ik}, \end{aligned}$$

where $\nu'_{ik} = \nu'_{ij} + \nu'_{jk}$ and $\nu''_{ik} = \nu''_{ij} + \nu''_{jk}$. □

Application of the first two kinds of syzygy reduces the number of relations per the following table

type 1,	type 2,
$f_{34} = f_{43}$	$g_{34} + g_{45} = g_{35}$
$f_{35} = f_{53}$	$g_{35} + g_{54} = g_{34}$
$f_{45} = f_{54}$	$g_{45} + g_{53} = g_{43}$
	$g_{43} + g_{35} = g_{45}$
	$g_{53} + g_{34} = g_{54}$
	$g_{54} + g_{43} = g_{53}$

which leaves only the following generators for the ideal

$$f_{34}, f_{35}, f_{45}, g_{34}, g_{35}, g_{45}$$

subject to the relations

$$\rho_{354}g_{34} = x_4 f_{35} - x_5 f_{34} + \sum_{\substack{l=3 \\ l\neq 4}} \alpha_{35l} f_{4l} - \sum_{\substack{l=3 \\ l\neq 4}} \alpha_{34l} f_{5l},$$

$$\rho_{345}g_{35} = x_5 f_{34} - x_4 f_{35} + \sum_{\substack{l=3 \\ l\neq 5}} \alpha_{34l} f_{5l} - \sum_{\substack{l=3 \\ l\neq 5}} \alpha_{35l} f_{4l},$$

and

$$\rho_{435}g_{45} = x_5 f_{43} - x_3 f_{45} + \sum_{\substack{l=3 \\ l\neq 5}} \alpha_{43l} f_{5l} - \sum_{\substack{l=3 \\ l\neq 5}} \alpha_{45l} f_{3l}.$$

It turns out the existence of the degree 3 equations characterizes some of Petri's exceptional cases per the following fact.

Lemma 1.4.
Let X be a genus 5 canonical, non-hyperelliptic, smooth, irreducible, complex algebraic curve. The sygygies of 1.3 satisfy the the following condition. Either $\rho_{ijk} = \alpha_{ijk} = 0$ whenever i, j, k are distinct, in which case C is either trigonal or in the genus 6 case may be a nonsingular plane

quintic, or $\{3, \cdots, g\} = I_1 \cup I_2$*, where for all* $j \in I_1$ *and* $k \in I_2$ *there exists an* i *with* $\rho_{ijk} \neq 0$ *and* $\alpha_{ijk} \neq 0$ *and such that the ideal of* C *is generated by the* f_{ij} *alone.*

Note that this latter case applies to the genus 5 curves in these notes and will be discussed more, but to be clear the trigonal map in the genus 5 case if all ρ vanish is given by the vanishings of the 3rd kind of syzygy, each a degree 1 map to $\mathbb{P}^1$ respectively.

Theorem 1.5. *Let* X *be a genus* 5 *canonical, non-hyperelliptic, smooth, irreducible, complex algebraic curve. The syzygies* f_{34}, f_{35} *and* f_{45} *generate the canonical ideal of* X *its canonical embedding in* $\mathbb{P}^4$*.*

Proof. Consider any partition of $\{3, 4, 5\}$ which includes at least one nonempty subset and the set theoretic complement of that the first component. If at least some $\rho_{ijk} \neq 0$ then g_{ik} is determined by the f_{ij}. If every g were to be 0 the result also follows. □

Because the object of interest for this inductive style argument is $\ker\left(H^0(X,\Omega)^{\oplus 2} \to H^0(X,\Omega^{\otimes 2})\right)$, a cohomological version of the proof is, forgive the pun, natural. It is worth noting that this choice of basis is not arbitrarily restrictive. Let $\varphi_1, \cdots, \varphi_g$ be a basis of differential forms for $H^0(C, \omega_C)$ such that

$$\begin{cases} \varphi_i(x_i) \neq 0 \\ \varphi_i(x_j) = 0, \quad \text{if } i \neq j. \end{cases}$$

Lemma 1.6. *Let* X *be a genus* 5 *canonical, non-hyperelliptic, smooth, irreducible, complex algebraic curve. Let* $\varphi : X \to \mathbb{P}^4$ *be the map obtained from global sections of the canonical bundle*

$$p \mapsto [s_1(p), \cdots, s_5(p)]$$

and let $x_1, \cdots, x_5 \in X$ *be some closed points in general position. Suppose* $\varphi_1, \cdots, \varphi_5$ *form a basis for* $H^0(X, \Omega)$ *such that* $\varphi_i(x_j) \neq 0$ *if and only if* $i = j$*. Given any basis* $\eta_1, \cdots, \eta_g$ *there exist some* $a_{i,j} \in \mathbb{C}$ *such that* $\varphi_i = \sum_{k=1}^{g} a_{i,k}\eta_k$*.*

Proof. Let $\eta_1, \cdots, \eta_g$ be a basis for $H^0(C, \Omega_C)$. Since the data of $H^0(C, \Omega_C)$ is some cover by affine opens $(U_i \to C)_{i \in \Lambda}$ with sections $s_i \in \Omega_C(U_i)$ compatible over intersections, for any $x \in C$, the η's globally generate $H^0(C, \Omega_C)$ in the sense that

$$\Omega_{C,x} = \operatorname{span}\{(\eta_1)_x, \cdots, (\eta_g)_x\}.$$

One of the rational sections $(\eta_i)_x$ is a generator for the localization $\Omega_{C,x}$ at x. Suppose for each of $x_1, \cdots, x_g \in C$ some closed points in general position, that $\alpha_1, \cdots, \alpha_g$ generate $\Omega_{C,x_1}, \cdots, \Omega_{C,x_g}$ respectively. Then

$$\begin{array}{llll} (\eta_1)_{x_1} = r_1\alpha_1 & (\eta_1)_{x_2} = r_2\alpha_2 & \cdots & (\eta_1)_{x_g} = r_g\alpha_g \\ (\eta_2)_{x_1} = s_1\alpha_1 & (\eta_2)_{x_2} = s_2\alpha_2 & \cdots & (\eta_2)_{x_g} = s_g\alpha_g \\ \vdots & & & \vdots \\ (\eta_g)_{x_1} = t_1\alpha_1 & & \cdots & (\eta_g)_{x_g} = t_g\alpha_g \end{array}$$

for some $r_1, s_1, \cdots, t_1 \in \mathcal{O}_{C,x_1},\ r_2, s_2, \cdots, t_2 \in \mathcal{O}_{C,x_2},\ r_g, s_g, \cdots, t_g \in \mathcal{O}_{C,x_g}$ and so on. Recall that each of the local rings $\mathcal{O}_{C,x_i}$ is a discrete valuation ring with a unique maximal ideal the uniformizer at x_i. Since Ω_{C,x_i} is generated by α_i for each i,

$$\langle r_i, s_i, \cdots, t_i \rangle = \mathcal{O}_{C,x_i}$$

so one of $r_i, s_i, \cdots, t_i \in \mathcal{O}_{C,x_i}^{\times}$. Suppose for some $a_{i,1}, \cdots, a_{i,g} \in \mathbb{C}$ not all 0 that

$$(a_{i,1}\eta_1 + a_{i,2}\eta_2 + \cdots + a_{i,g}\eta_g)(x_j) = 0$$

for some $j \neq i$. At the stalk

$$(a_{i,1}\eta_1 + a_{i,2}\eta_2 + \cdots + a_{i,g}\eta_g)_{x_j} = [a_{i,1}(r_j(x_j)) + a_{i,2}(s_j(x_j)) + \cdots + a_{i,g}(t_j(x_j))]\,\alpha_j$$

so without loss of generality if r_j is the unit, since $a_{i,1}r_j + \cdots + a_{1,g}t_j = 0$,

$$a_{i,1} = -r_j^{-1}(x_j)\left[a_{i,2}s_j(x_j) + \cdots + a_{i,g}t_j(x_j)\right].$$

In particular the solution lies in $\mathbb{C}$. Indeed $r_j, s_j, \cdots, t_j \in \mathcal{O}_{C,x_j}$ so the evaluations $r_j(x_j), \cdots, s_j(x_j) \in \mathcal{O}_{C,x_j}/\mathcal{M} = \kappa(C)$, where $\mathcal{M}$ is the uniformizer at x_2 and $\kappa(C) = \kappa(x_j)$ is the residue field of the curve at the stalk. So since $s_j, \cdots, t_j$ vanish to nonnegative order at x_j as localizations of a global section to an affine open, and r_j by assumption of being a unit is nonvanishing at x_j,

$$r_j \in \mathcal{O}_{C,x_j}^{\times} \Rightarrow r_j(x_j) \in (\mathcal{O}_{C,x_j}/\mathcal{M})^{\times} = \mathbb{C}^{\times}$$

and each of $s_j(x_j), \cdots, t_j(x_j)$ lie in a finite extension of $\mathbb{C}$, hence each is a complex number since $\mathbb{C}$ is algebraically closed, so there are no such nontrivial extensions of $\mathbb{C}$. $\square$

Finally, one last word of introduction is in order here. It is an exercise level problem for modern mathematicians to prove that the complete intersection of 3 quadrics in $\mathbb{P}^4$ is a genus 5 curve. The main ideas are to use the adjunction formula and the corresponding genus formula. If Y is the complete intersection of the degrees d_1, d_2, d_3 hypersurfaces $D_1, D_2, D_3 \subset \mathbb{P}^4$ respectively, then

$$g_y = \frac{(d_1 + d_2 + d_3 - 5)d_1d_2d_3 + 2}{2}.$$

Therefore, it is the purpose of these notes to go the other way, namely from the curve to the complete intersection which defines it in its canonical embedding.

2. Background

This section has parts which resectively introduce facts about line bundles, states the sheaf relation given by the Euler sequence, along with relevant twists, and defines the Koszul cohomology. All of this is done with the specific example of a genus g canonical, non-hyperelliptic, smooth, irreducible complex algebraic curve in mind. Line bundle facts come from [18], [19], [20] and [21] and Section 2.1 is about the properties with which a line bundle gives an embedding and the properties of the embedded curve. The Euler sequence is from [25] while the twists and pullbacks come from [10] with the Stacks Project references above, and Section 2.2 states Euler's relation between the sheaf of differentials and the structure sheaf along with the twists and wedge products needed later for Koszul cohomology. Finally Koszul cohomology is defined in [11], used in [10], and the relevant information about wedge products and explicit maps comes from [5], so Section 2.3 concretely defines the Koszul complex and motivates why it is relevant to the discussion of Petri's equations. The first two sections are mostly standard facts but the selection of those facts used for this problem is not from any particular source.

2.1. **Line Bundle Facts.**

The main object of study in these notes is the following object.

Definition 2.1. [12, 1.2] *For a line bundle L on a scheme X the section ring is the graded ring*

$$R(X, L) = \bigoplus_{d \in \mathbb{N}} H^0(X, L^{\otimes d})$$

and is also called the Cox ring.

A necessary condition for a line bundle to give an embedding as the canonical bundle does, has to do with the generation of that invertible sheaf L.

Definition 2.2. [10] *Say a line bundle L on X a scheme is globally generated or generated by its global sections if there is an inclusion*

$$\begin{array}{rcl} \varphi_L : X & \hookrightarrow & \mathbb{P}(H^0(X, L)) = \mathbb{P}^r \\ \Psi & & \Psi \\ p & \mapsto & [s_0(p), \cdots, s_r(p)]. \end{array}$$

The other necessary condition for a line bundle to embed with as good of properties as the canonical embedding involves the common vanishing of global sections.

Definition 2.3. *The base locus of a line bundle L on X is*

$$\bigcap_{s \in H^0(X,L)} \{s = 0\}.$$

Say that L is basepoint free if the base locus is empty

In particular, the way a line bundle gives an embedding of a variety into projective space can be stated with the following definitions.

Definition 2.4. [22] *Say the line bundle L is ample if there is some nonnegative $r \in \mathbb{Z}$ such that $L^{\otimes r}$ is very ample.*

Definition 2.5. [22] *Say a line bundle L is very ample if the embedding $\varphi_L : X \to \mathbb{P}^r$ by global sections of L is a closed immersion and L is basepoint free.*

Turning from the embedding to the embedded object, the corresponding property to global generation is the normality of the embedded subvariety.

Definition 2.6. [11] *Let B be a ring. Some subvariety $V \subset \mathbb{P}^r_B$ is projectively normal if the canonical maps*

$$H^0(\mathbb{P}^r_B, \mathcal{O}_{\mathbb{P}^r_B}(d)) \to H^0(V, \mathcal{O}_V(d)),$$

where $\mathcal{O}_V \cong \mathcal{O}_{\mathbb{P}^r}/I_V$ is the structure sheaf on V, are surjective for all $d > 0$.

The major result in this section and the biggest theorem as such which is proved in this document is Max Noether's theorem. To set that up requires just one more definition and one intermediate fact about terminology.

Definition 2.7. [10] *Say the line bundle L is normally generated if the maps $\rho_k : \operatorname{Sym}^k H^0(L) \to H^0(L^k)$ are surjective for all $k \geqslant 0$.*

Fact 2.8. *L is a normally generated line bundle if and only if the embedded curve $\varphi_L(X) \subset \mathbb{P}^r$, for $r = h^0(L) - 1$ is projectively normal.*

Proof. Under the identification $\mathcal{O}_{\mathbb{P}^r_B}(d) = \widetilde{S_{\mathbb{P}^r_B}(d)}$,

$$H^0(\mathbb{P}^r_B, \mathcal{O}_{\mathbb{P}^r_B}(d)) = B[x_0, \cdots, x_r]_d,$$

and since $H^0(X, \varphi_L^* \mathcal{O}_X(d)) = H^0(X, L^{\otimes d})$ and

$$\operatorname{Sym}^k(H^0(X, L)) = \operatorname{Sym}^k(Bs_0 \oplus \cdots \oplus Bs_r) = R[s_0, \cdots, s_r]_k,$$

where $R = R(L)$ is the Cox ring of the line bundle, and everything happens over the base ring B, the definitions of normally generated and projectively normal are the same. ☐

The intersection theory result which is the title of these notes is made possible thanks to Noether's theorem which is stated in one form here. The proof is reserved for a later section of its own.

Theorem 2.9 (Noether)**.** [10] *A canonically embedded nonhyperelliptic curve X with genus g is projectively normal. That is to say the maps*

$$H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(k)) \to H^0(X, \Omega_X^k)$$

are surjective for all $k \geqslant 0$.

For clarity, these facts are connected by the following statement.

Lemma 2.10.
Suppose L is a line bundle on a scheme X.

(1) *If L is very ample then it is basepoint free.*
(2) *L is basepoint free if and only if it is globally generated.*

Proof. If L is very ample, the induced map $\phi_L : X \to \mathbb{P}^r$ is a closed immersion. But L is basepoint free if and only if there exists a morphism $\phi_L : X \to \mathbb{P}^r$ and that map is already given. If L is basepoint free then there is no $x \in X$ such that for all $s \in H^0(X, L)$, $s_x \in \mathcal{M}_x L_x$ since $s(x) = 0$ if and only if $s_x \equiv 0 \pmod{\mathcal{M}_X L_x}$ but the base locus of a basepoint free map is empty. If L is globally generated, for each $x \in X$ there is $s \in H^0(X, L)$ such that $\mathcal{O}_{X,x} s_x = L_x$, so L basepoint free. Therefore L is globally generated if and only if it is basepoint free. □

The last result in this section, from Serre, will come up while computing the cohomology in the Koszul complex section.

Theorem 2.11 (Serre vanishing)**.** [11, 2.a.6] *If L is an ample line bundle on X and $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules then*

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(qL)) = 0,\ q \ll 0,$$

where $q \ll 0$ means q is a sufficiently large negative number.

2.2. **Euler Sequence.** In this section, some relations between different sheaves are introduced. Eventually, by forming the long exact sequences in sheaf cohomology from different short exact sequences given here, the Koszul cohomology will be able to inductively demonstrate that all syzygies for the canonical ideal are generated in degree 2.

This is one of the most useful definitions in this document and will come up many times.

Definition 2.12. *The Euler sequence on $\mathbb{P}^n$ is the following exact sequence of sheaves on $\mathbb{P}^n$*

$$0 \to \Omega_{\mathbb{P}^n_A} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0$$

which relates the sheaf of holomorphic differentials Ω to the structure sheaf on $\mathcal{O}_{\mathbb{P}^n}$.

For rigor, it is worth checking exactness.

Lemma 2.13. [24] *The Euler sequence is exact.*

Proof. Let $\varphi : \mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O}$ be the degree 1 map

$$(s_0, \cdots, s_n) \mapsto x_0 s_0 + \cdots + x_n s_n.$$

Identifying the kernel of this map with differentials can be done locally since injectivity and surjectivity are local properties. Consider U_0 where $x_0 \neq 0$ some open set. Consider some coordinates $x_{j/0} = \frac{x_j}{x_0}$ for $1 < j \leqslant n$. To each differential

$$f_1(x_{1/0}, x_{2/0}, \cdots, x_{n/0}) dx_{1/0} + \cdots + f_n(x_{1/0}, \cdots, x_{n/0}) dx_{n/0} \in \Omega_{\mathbb{P}^n}$$

there are $n+1$ sections of $\mathcal{O}(-1)$ since by treating the projective coordinates naively,

$$\begin{aligned} f_1 dx_{1/0} &= f_1 d(\tfrac{x_1}{x_0}) \\ &= f_1 \tfrac{x_0 dx_1 - x_1 dx_0}{x_0^2} \\ &= \tfrac{f_1}{x_0} dx_1 + \tfrac{-x_1}{x_0^2} f_1 dx_0. \end{aligned}$$

Note $x_0 \left(\frac{-x_1}{x_0^2} f_1\right) + x_1 \left(\frac{f_1}{x_0}\right) = 0$ and that both $\frac{-x_1}{x_0^2} f_1$ and $\frac{f_1}{x_0}$ are homogeneous of degree -1.
Let $\imath : \Omega_{\mathbb{P}^n_A} \to \mathcal{O}(-1)^{\oplus(n+1)}$ be given by

$$f_1 dx_{1/0} + \cdots + f_n dx_{n/0} \mapsto \left(-\frac{x_1}{x_0^2} f_1 - \cdots - \frac{x_n}{x_0^2} f_n, \frac{f_1}{x_0}, \frac{f_2}{x_0}, \cdots, \frac{f_n}{x_0}\right).$$

First of all $\imath|_{U_0}(\Omega_{\mathbb{P}^n}) \subseteq \ker \varphi$ since

$$x_0 \left(-\frac{x_1}{x_0^2} f_1 - \cdots - \frac{x_n}{x_0^2} f_n\right) + x_1 \left(\frac{f_1}{x_0}\right) + \cdots + x_n \left(\frac{f_n}{x_0}\right) = 0$$

Then $\imath|_{U_0}$ is one-to-one since $\ker \imath|_{U_0} = \{0\}$ as $\imath(f_1 dx_{1/0} + \cdots + f_n dx_{n/0}) = (0, \cdots, 0)$ if and only if $f_i = 0$ for $1 \leqslant i \leqslant n$.
Also $\imath$ surjects onto the kernel of $\mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O}_X$ since for

$$(g_0, \cdots, g_n) \in \ker(\mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O}_X),$$

let $f_i = x_0 g_i$ for each $1 \leqslant i \leqslant n$. To verify this construction consider the map on two different coordinate patches at once, say $U_0 \cap U_1$, where in particular there should be a compatible solution. Note that

$$\begin{aligned} f_1 dx_{1/0} + f_2 dx_{2/0} + \cdots + f_n dx_{n/0} &= f_1 d\tfrac{1}{x_{0/1}} + f_2 d\tfrac{x_{2/1}}{x_{0/1}} + \cdots + f_n d\tfrac{x_{n/1}}{x_{0/1}} \\ &= \tfrac{-f_1}{x_{0/1}^2} dx_{0/1} + \tfrac{x_{0/1}dx_{2/1} - x_{2/1}dx_{0/1}}{x_{0/1}^2} f_2 + \cdots + \tfrac{x_{0/1}dx_{n/1} - x_{n/1}dx_{0/1}}{x_{0/1}^2} f_n \\ &= \tfrac{-f_1}{x_{0/1}^2} dx_{0/1} + \tfrac{f_2}{x_{0/1}} dx_{2/1} - \tfrac{f_2 x_{2/1}}{x_{0/1}^2} dx_{0/1} + \cdots + \tfrac{f_n}{x_{0/1}} dx_{n/1} - \tfrac{f_n x_{n/1}}{x_{0/1}^2} dx_{0/1} \\ &= -\tfrac{f_1 + f_2 x_{2/1} + \cdots + f_n x_{n/1}}{x_{0/1}^2} dx_{0/1} + \tfrac{f_2}{x_{0/1}} dx_{2/1} + \cdots + \tfrac{f_n}{x_{0/1}} dx_{n/1} \\ &= -\tfrac{f_1 + f_2 x_{2/1} + \cdots + f_n x_{n/1}}{x_{0/1}^2} dx_{0/1} + \tfrac{f_2 x_1}{x_0} dx_{2/1} + \cdots + \tfrac{f_n x_1}{x_0} dx_{n/1}. \end{aligned}$$

In particular the $dx_{2/1}$ term maps to the second factor in $\mathcal{O}(-1)^{\oplus(n+1)}$ and gives $\frac{f_2}{x_0}$ as desired and likewise for each $dx_{j/1}$ term for $j > 2$. Also the $dx_{0/1}$ term goes to zero factor

$$\frac{\left(\sum_{j=1}^n f_i \frac{(x_i/x_1)}{(x_0/x_1)^2}\right)}{x_1} = f_1 \frac{x_i}{x_0^2}$$

as desired. The first factor must be corrected because the $\sum_i x_i(i\text{th factor}) = 0$. □

Twists of the Euler sequence will show up in a few different forms, but the first one, below, will be the form of the Euler sequence most used in these notes.

Lemma 2.14. *Since the Euler sequence is exact then the following is exact*

$$0 \to \Omega_{\mathbb{P}^n}(1) \to \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n}(1) \to 0, \tag{2.1}$$

Now the stage can be set for the Koszul complex with some manipulation of the Euler sequence from the previous lemma.

Lemma 2.15. *Let $L = \mathcal{O}_{\mathbb{P}^n}(1) \otimes_{\mathcal{O}_X} \mathcal{O}_X$. Let $r = h^0(L) - 1$ and $M_L = \varphi_L^* \Omega_{\mathbb{P}^r}(1)$. Then the following pullback by ϕ_L of the sequence above is exact*

$$0 \to M_L \to H^0(L) \otimes_{\mathbb{C}} \mathcal{O}_X \to L \to 0.$$

In this lemma, the pullback from before will get a twist.

Lemma 2.16. *The following is exact*

$$0 \to M_L \otimes L^{k-1} \to H^0(L) \otimes_{\mathbb{C}} L^{k-1} \to L \otimes L^{k-1} \to 0.$$

Proof. Since L is a line bundle it is locally free and hence twisting by L preserves exactness. Twist the sequence above in 2.15 by L^{k-1} to obtain the sequence in the statement. □

This lemma is just another twist, but this time with some wedge products.

Lemma 2.17. *The following is exact.*

$$0 \to \bigwedge^2 M_L \otimes L^{k-1} \to \bigwedge^2 H^0(L) \otimes_{\mathbb{C}} L^{k-1} \to M_L \otimes L^k \to 0. \tag{2.2}$$

Proof. Taking wedge products in Lemma 2.15 and twisting by L^{k-1} also preserves exactness so to obtain the sequence in the statement, first consider the dual sequence

$$0 \to L^\vee \to H^0(L)^\vee \otimes_{\mathbb{C}} \mathcal{O}_X \to M_L^\vee \to 0.$$

By [18, Tag 00DM] the following is exact

$$L^\vee \otimes H^0(L)^\vee \otimes \mathcal{O}_X \to \bigwedge^2 H^0(L)^\vee \otimes \mathcal{O}_X \to \bigwedge^2 M_L^\vee \to 0.$$

Take the dual again, note that $M_L \cong L \otimes_{\mathcal{O}_X} H^0(L) \otimes_{\mathbb{C}} \mathcal{O}_X$ by Lemma 2.15, and the following is exact

$$0 \to \bigwedge^2 M_L \to \bigwedge^2 H^0(L) \otimes_{\mathbb{C}} \mathcal{O}_X \to M_L$$

and twisting by L^{k-1} finally gives

$$0 \to \bigwedge^2 M_L \otimes L^{k-1} \to \bigwedge^2 H^0(L) \otimes_{\mathbb{C}} L^{k-1} \to M_L \otimes L^k.$$

The rightmost map is given by

$$(s_1 \wedge s_2) \otimes f \mapsto s_1 \otimes s_2 f - s_2 \otimes s_1 f$$

and is surjective since this is a Koszul map $d_{2,k-1}$ composed of $(\mathrm{Id} \otimes m_{k-1})$ and $(\psi_{\mathrm{id}} \otimes \mathrm{Id})$, where ψ_{id} is dual to an injective map and is surjective, and m_{k-1} is surjective by definition of the multiplication map in $\oplus_{k \in \mathbb{N}} L^k$. This makes the sequence right exact. □

2.3. **Koszul Complex.**

In this section, the Koszul cohomology is introduced. This tool will give an interpretation of Petri's theorem, which this document is an example of, as a statement about cohomology. The actual computational technique which come from this complex is stated in Theorem **??**, but this section is focused on demonstrating what kinds of maps exist in the Koszul complex, and that it is indeed a well-defined complex.

Let $\mathbb{F}$ be a field, let V be an n-dimensional $\mathbb{F}$-vector space and let $B = \bigoplus_{q \in \mathbb{Z}} B_q$ be a graded $\mathrm{Sym}(V)$-module. Note that the abstract looking $\mathrm{Sym}(V)$ in the context of the embedded curves in these notes is actually quite a familiar ring, but for greater generality in future projects the result will be phrased with slightly different notation.

Fact 2.18. *Let R be a ring and let M be a free R-module with basis $y_0, \cdots, y_n$. The homogeneous coordinate ring of $\mathbb{P}^n_R$ is* $\mathrm{Sym}(M) \cong R[y_0, \cdots, y_n]$.

Proof. Both the symmetric algebra $\mathrm{Sym}(M)$ and the polynomial ring $R[y_0, \cdots, y_n]$, where the y_i are a basis are free objects in their respective categories. The homogeneous polynomials of degree 1 are a free R-module which can be identified with M itself and in particular satisfies the following universal property of the symmetric algebra: for every linear $f : M \to A$ a morphism of algebras, there is a unique algebra homomorphism $g : \mathrm{Sym}(M) \to A$ such that $f = g \circ i$, for $i : M \to \mathrm{Sym}(M)$ the inclusion map. Suppose that $f' : R[y_0, \cdots, y_n]_1 \to A$ is a linear algebra morphism for some R-algebra A. Then since $R[y_0, \cdots, y_n]$ is the free object in the category of R-algebras there is the unique $g' : R[y_0, \cdots, y_n] \to A$ such that $f' = g' \circ i'$ for $i' : R[y_0, \cdots, y_n]_1 \hookrightarrow R[y_0, \cdots, y_n]$. □

For the sake of the notation used in the Koszul complex in these notes the coordinate ring will still be written $\mathrm{Sym}(V)$ since this will make some nontrivial identifications easier to see once the more intricate proofs begin. But for intuition it is nice to keep in mind that the symmetric algebra in this context is nothing more than a typical polynomial ring. It is also advantageous to phrase as many definitions and proofs for the Koszul complex in terms of the symmetric algebra since it functions as a coordinate free version of the polynomial ring per 2.18. With all of this set up, it is time to define the Koszul cohomology.

Definition 2.19 (Koszul complex)**.** [11, 1.a.2] *Let $\mathbb{F}$ be a field, let V be an n-dimensional $\mathbb{F}$-vector space and let $B = \bigoplus_{q\in\mathbb{Z}} B_q$ be a graded* $\mathrm{Sym}(V)$*-module. The Koszul complex is the long exact sequence*

$$\cdots \to \bigwedge^{p+1} V \otimes B_{q-1} \overset{d_{p+1,q-1}}{\to} \bigwedge^{p} V \otimes B_{q} \overset{d_{p,q}}{\to} \bigwedge^{p-1} V \otimes B_{q+1} \overset{d_{p-1,q+1}}{\to} \bigwedge^{p-2} V \otimes B_{q+2} \overset{d_{p-2,q+2}}{\to} \cdots$$

where the maps $d_{p,q}$ are defined to be the composite maps

$$d_{p,q} = (\mathrm{Id} \otimes m_q) \circ (\psi_{\mathrm{id}} \otimes \mathrm{Id}),$$

$$\textit{where} \begin{cases} \psi_{\mathrm{id}} : \bigwedge^p V \to \bigwedge^{p-1} V \otimes V, & \textit{is dual to the exterior product map} \\ m_q : V \otimes B_q \to B_{q+1}, & \textit{is multiplication in } B, \end{cases}$$

such that the following commutes

$$\begin{array}{ccc} \bigwedge^p V \otimes B_q & \xrightarrow{\psi_{\mathrm{id}}\otimes \mathrm{Id}} & \bigwedge^{p-1} V \otimes V \otimes B_q \\ & \searrow^{d_{p,q}} & \downarrow \mathrm{Id} \otimes m_q \\ & & \bigwedge^{p-1} V \otimes B_{q+1} \end{array}$$

Each of these maps and the commutativity of the diagram is worked out explicitly in this section. First of all, the Koszul cohomology groups need to be defined.

Definition 2.20 (Koszul cohomology groups)**.** [11, 1.a.7] *Let $\mathbb{F}$ be a field, let V be an n-dimensional $\mathbb{F}$-vector space and let $B = \bigoplus_{q\in\mathbb{Z}} B_q$ be a graded* $\mathrm{Sym}(V)$*-module. The Koszul cohomology groups of B are the groups*

$$K_{p,q}(B, V) = \frac{\ker d_{p,q}}{\mathrm{im}\, d_{p+1,q-1}}.$$

It turns out that the Koszul complex terminates after finitely many maps in either direction, which is established by the next definition.

Definition 2.21 (Koszul conventions)**.** [11, 1.a.8] *Let $\mathbb{F}$ be a field, let V be an n-dimensional $\mathbb{F}$-vector space and let $B = \bigoplus_{q\in\mathbb{Z}} B_q$ be a graded* $\mathrm{Sym}(V)$*-module. Say $K_{p,q}(B, V) = 0$ when $p < 0$ or $p > \dim V$.*

Likewise with the Cox ring being a fundamental object of study in these notes, this next definition is the formal way to state what the Koszul complex is able compute.

Example. If $x_1, x_2, \cdots$ are generators for B with $\deg x_i = e_i$ then a weight q relation among the generators has form

$$\sum_i u_i x_i, \;\; u_i \in \mathrm{Sym}^{q-e_i}(V),$$

and a primitive relation is one which is not a $\mathrm{Sym}(V)$ linear combination of relations of lower weight. If $\sum_i u_i^\nu x_i$ are a basis of primitive relations of weights e^ν respectively, a weight q syzygy is

$$\sum_\nu w_\nu u_i^\nu = 0 \text{ for all } i, \;\; w_\nu \in \mathrm{Sym}^{q-e^\nu}(V).$$

Definition 2.22. [11, 1.b.3] *Let $\mathbb{F}$ be a field, let V be an n-dimensional $\mathbb{F}$-vector space and let $B = \bigoplus_{q\in\mathbb{Z}} B_q$ be a graded* $\mathrm{Sym}(V)$*-module. The* syzygies *of order p and weight q of B form a* $\mathrm{Sym}(V)$*-module denoted $M_{p,q}(B,V)$.*

The definition of order p and weight q syzygies is inductive. Say that
$M_{0,q}$ is the module of degree q generators for B as a $\mathrm{Sym}(V)$-module,
$M_{1,q}$ is the module of primitive relations in weight q for B,
$M_{2,q}$ is the module of syzygies of weight q among relations for B and so on.

The computation by Koszul groups of these syzygies comes from this theorem of Green.

Theorem 2.23 (Syzygy)**.** [11, 1.b.4] *Let $\mathbb{F}$ be a field, let V be an n-dimensional $\mathbb{F}$-vector space and let $B = \bigoplus_{q\in\mathbb{Z}} B_q$ be a graded* $\mathrm{Sym}(V)$*-module.*
$K_{p,q}(B,V) \cong M_{p,p+q}(B,V)$ as $\mathbb{F}$-vector spaces.

Now that the premilinary definitions are stated, and the motivation for considering the Koszul complex is hidden in Theorem **??**, the promised explicit description of the complex follows.

Let N be an R-module, and $\varphi : N \to R$.
Consider a diagonalization $\Delta : \bigwedge N \to \bigwedge N \otimes_R \bigwedge N$ the unique map of algebras defined by

$$m \mapsto m \otimes 1 + 1 \otimes m$$

for $m \in \bigwedge^1 N = N$ and $m \otimes 1 + 1 \otimes m \in \bigwedge N \otimes \bigwedge^0 N \oplus \bigwedge^0 N \otimes \bigwedge N \subset \bigwedge N \otimes \bigwedge N$.
In particular the component of Δ which maps $\bigwedge^i N \to N \otimes \bigwedge^{i-1} N$ given on generators by

$$\Delta'(m_1 \wedge \cdots \wedge m_i) = \sum_{j=1}^{i} (-1)^{j-1} m_j \otimes m_1 \wedge \cdots \wedge \hat{m_j} \wedge \cdots \wedge m_i,$$

where $\hat{m_j}$ means that m_j is left out of the product, gives a description of the differentials

$$\delta_\varphi : \bigwedge^{i} N \to \bigwedge^{i-1} N.$$

Define δ_φ to be the composite

$$\bigwedge^{i} N \xrightarrow{\Delta'} N \otimes_R (\bigwedge^{i-1} N) \xrightarrow{\varphi\otimes 1} R \otimes_R \bigwedge^{i-1} N = \bigwedge^{i-1} N.$$

Note when $i = 1$ the composite is just φ. Then $\delta_\varphi^2(n_1 \wedge \cdots \wedge n_i)$ is a linear combination of terms $n_1 \wedge \cdots \wedge \hat{n_j} \wedge \cdots \wedge \hat{n_{j'}} \wedge \cdots \wedge n_i$. If $j < j'$ then the coefficient of the term above in $\delta_\varphi^2(n_1 \wedge \cdots \wedge n_i)$ is

$$(-1)^j(-1)^{j'-1}\varphi(n_j)\varphi(n_{j'}) + (-1)^j(-1)^{j'}\varphi(n_j)\varphi(n_{j'}) = 0$$

so $\delta_\varphi^2 = 0$.

Lemma 2.24. *Let $V/\mathbb{F}$ be a finite dimensional vector space and* $\mathrm{Sym}(V)$ *be the symmetric algebra over V. Then the differentials $\delta_\varphi : \bigwedge^p V \to \bigwedge^{p-2} V$ for $\varphi \in V^*$ satisfy $\delta_\varphi^2 = 0$.*

Proof. Let $\Delta : \bigwedge V \to \bigwedge V$ be the map $x \mapsto x \otimes 1 + 1 \otimes x$. Consider $\Delta' : \bigwedge^p V \to V \otimes \bigwedge^{p-1} V$ given on the basis by

$$\Delta'(m_1 \wedge \cdots \wedge m_p) = \sum_{j=1}^{p} (-1)^{j-1} m_j \otimes m_1 \wedge m_2 \wedge \cdots \wedge \hat{m_j} \wedge \cdots \wedge m_p.$$

Let $\varphi \in V^* = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ and let $\delta_\varphi := (\varphi \otimes 1) \circ \Delta'$ be the composite map

$$\bigwedge^p V \xrightarrow{\Delta'} V \otimes (\bigwedge^{p-1} V) \xrightarrow{\varphi \otimes 1} \bigwedge^{p-1} V.$$

Then

$$\begin{aligned} \delta_\varphi(m_1 \wedge \cdots \wedge m_p) &= (\varphi \otimes 1)(\textstyle\sum_{j=1}^p (-1)^{j-1} m_j \otimes m_1 \wedge \cdots \wedge \hat{m}_j \wedge \cdots \wedge m_p) \\ &= \varphi(\textstyle\sum_{j=1}^p (-1)^{j-1} m_j) \otimes m_1 \wedge \cdots \wedge \hat{m}_j \wedge \cdots \wedge m_p \\ &= \textstyle\sum_{j=1}^p (-1)^{j-1} \varphi(m_j) \otimes m_1 \wedge \cdots \wedge \hat{m}_j \wedge \cdots \wedge m_p \end{aligned}$$

and extend by linearity, so

$$\begin{aligned} \delta_\varphi^2(m_1 \wedge \cdots \wedge m_p) &= \delta_\varphi(\textstyle\sum_{j=1}^p (-1)^{j-1} \varphi(m_j) \otimes m_1 \wedge \cdots \wedge \hat{m}_j \wedge \cdots \wedge m_p) \\ &= (\varphi \otimes 1)(\textstyle\sum_{k=1}^p (-1)^{k-1} m_k \otimes \sum_{j=1}^p (-1)^{j-1} \varphi(m_j) \otimes m_1 \wedge \cdots \wedge \hat{m}_j \wedge \cdots \wedge \hat{m}_k \wedge \cdots \wedge m_p) \\ &= \textstyle\sum_{k=1}^p (-1)^{k-1} \varphi(m_k) \otimes \sum_{j=1}^p (-1)^{j-1} \varphi(m_j) \otimes m_1 \wedge \cdots \wedge \hat{m}_j \wedge \cdots \wedge \hat{m}_k \wedge \cdots \wedge m_p). \end{aligned}$$

Since a basis for $\bigwedge^{p-2} V$, a free rank $\binom{\dim_{\mathbb{F}}(V)}{p-2}$ $\mathbb{F}$-module, is

$$\{v_{i_1}, \cdots, v_{i_{p-2}} : 1 \leqslant i_1 < \cdots < 1_{p-2} \leqslant \dim_{\mathbb{F}}(V)\}$$

corresponding to all $(p-2)$-subsets of $\{1, \cdots, \dim_K(V)\}$, we can write

$$\delta_\varphi^2(m_1 \wedge \cdots \wedge m_p) = \sum_{l=0}^{\binom{\dim_{\mathbb{F}}(V)}{p-2}} a_l (v_{l_1} \wedge \cdots \wedge v_{l_{p-2}})$$

where the a_l of the term $m_1 \wedge \cdots \wedge \hat{m}_j \wedge \cdots \wedge \hat{m}_k \wedge \cdots \wedge m_p)$ is

$$(-1)^{k-2}(-1)^{j-1}\varphi(m_k)\varphi(m_j) + (-1)^{k-1}(-1)^{j-1}\varphi(m_k)\varphi(m_j) = 0,$$

since for each interchange of m_i and m_j in the wedge to bring m_j out in front, or to leave it out in Eisenbud's terminology, a factor of -1 is added, conclude that $\delta_\varphi^2 = 0$. □

Lemma 2.25. *Let $V/\mathbb{F}$ be a finite dimensional vector space with basis $v_1, \cdots, v_n$. Then the dual to the exterior product map*

$$V^\vee \wedge \bigwedge^{p-1} V^\vee \to \bigwedge^p V^\vee$$

given by

$$v^\vee \otimes \alpha \mapsto v^\vee \wedge \alpha$$

is the component Δ' of the diagonal map on the pth graded piece $\bigwedge^P V$ of the exterior algebra $\bigwedge V$ given by

$$\Delta'(v_1 \wedge \cdots \wedge v_p) = \sum_{j=1}^p (-1)^{j-1} v_j \otimes v_1 \wedge \cdots \wedge \hat{v_j} \wedge \cdots \wedge v_p.$$

Proof. Consider the following diagram

$$\begin{array}{ccccccc} \alpha \wedge v^\vee & \in & \bigwedge^p V^\vee & \xleftrightarrow{\ \vee\ } & \bigwedge^p V & \ni & v_1 \wedge \cdots \wedge_p \\ \updownarrow & & \uparrow & & \downarrow & & \updownarrow \\ \alpha \otimes v^\vee & \in & \bigwedge^{p-1} V^\vee \wedge V^\vee & \xleftrightarrow{\ \vee\ } & \bigwedge^{p-1} V \otimes V & \ni & \Delta'(v_1 \wedge \cdots \wedge v_p) \end{array}$$

Since $(\bigwedge^p V)^\vee = \bigwedge^p V^\vee$ for $v_1^\vee \wedge \cdots \wedge v_p^\vee \in \bigwedge^p V^\vee$, $(v_1^\vee \wedge \cdots \wedge v_p^\vee)^\vee = v_1 \wedge \cdots \wedge v_p$, which is abbreviated v. Recall that

$$\Delta'(v) = \sum_{j=1}^p (-1)^{j-1} v_j \otimes v_1 \wedge \cdots \wedge \hat{v_j} \wedge \cdots \wedge v_p$$

so

$$(\Delta'(v))^\vee = \sum_{j=1}^{p}(-1)^{j-1}(v_j \otimes v_1 \wedge \cdots \wedge \hat{v_j} \wedge \cdots \wedge v_p)^\vee = \sum_{j=1}^{p}(-1)^{j-1} v_j^\vee \otimes v_1^\vee \wedge \cdots \wedge \hat{v_j} \wedge \cdots \wedge v_p^\vee$$

and applying the exterior product map $v^\vee \otimes \alpha \mapsto \alpha \wedge v^\vee$ to $\Delta'^\vee$ therefore yields

$$\sum_{j=1}^{p}(-1)^{j-1} v_j^\vee \wedge v_1^\vee \wedge \cdots \wedge \hat{v_j} \wedge \cdots \wedge v_p^\vee = v_1^\vee \wedge \cdots \wedge v_p^\vee$$

so indeed the dual to the exterior product map on duals given by

$$(\alpha \otimes v^\vee \mapsto \alpha \wedge v^\vee)$$

is Δ'. □

One pedagogical word. After a nice example at PCMI 2022 by Herny Cohn about how to define duals, the approach here is to offer both a definition that is clearly a dual object per the discussion following Theorem 2.23, and then to offer another definition which makes rigorous the well-definedness of the dual. This is a relationship between the exterior product map and the contraction-by-the-identity that will both verify that the Koszul complex is indeed a complex and be a computational tool used later on.

Let $\mathbb{F}$ be a field, let V be an n-dimensional $\mathbb{F}$-vector space and let B be a graded $\mathrm{Sym}(V)$-module. Suppose $\mathrm{id} \in V^\vee \otimes V$ is the identity. Note that $V^\vee \otimes V \cong \mathrm{End}(V)$ and $\mathrm{id} = \sum_{i=1}^n v_i^\vee \otimes v_i$. Since $v_i^\vee(v_j) = \delta_{i,j}$, the Kronecker delta, if $w = \sum_{j=1}^n c_j v_j \in V$ then

$$\begin{aligned} \mathrm{id}(w) &= (\textstyle\sum_{i=1}^n v_i^\vee \otimes v_i)(w) \\ &= \textstyle\sum_{i=1}^n v_i v_i^\vee(w) \\ &= \textstyle\sum_{i=1}^n v_i(\sum_{j=1}^n c_j v_i^\vee(v_j)) \\ &= \textstyle\sum_{i=1}^n v_i c_i. \end{aligned}$$

With this setup, the relationship mentioned above follows.

Lemma 2.26. *Let $\mathbb{F}$ be a field, let V be an n-dimensional $\mathbb{F}$-vector space and let $B = \bigoplus_{q\in\mathbb{Z}} B_q$ be a graded $\mathrm{Sym}(V)$-module. The dual to the exterior product map $\bigwedge^{p-1} V^\vee \otimes V^\vee \to \bigwedge^p V^\vee$ is a contraction-by-*id *map $\psi_{\mathrm{id}} : \bigwedge^p V \to \bigwedge^{p-1} V \otimes V$.*

Proof. Write $m_q : V \otimes B_q \to B_{q+1}$ for the multiplication map. Then define $d_{p,q}$ such that the following commutes

$$\begin{array}{ccc} \bigwedge^p V \otimes B_q & \xrightarrow{\psi_{\mathrm{id}}\otimes \mathrm{Id}} & \bigwedge^{p-1} V \otimes V \otimes B_q \\ & \searrow^{d_{p,q}} & \downarrow{\scriptstyle \mathrm{Id}\otimes m_q} \\ & & \bigwedge^{p-1} V \otimes B_{q+1} \end{array}$$

In particular, $\psi_{\mathrm{id}} = \Delta'$ by Claim 2.25 so $d_{p,q} = \Delta' \otimes m_q$. □

Finally, to check that $d^2 = 0$ in the Koszul complex observe that since $\delta_\varphi^2 = 0$, $d^2 = 0$ in this Koszul complex.

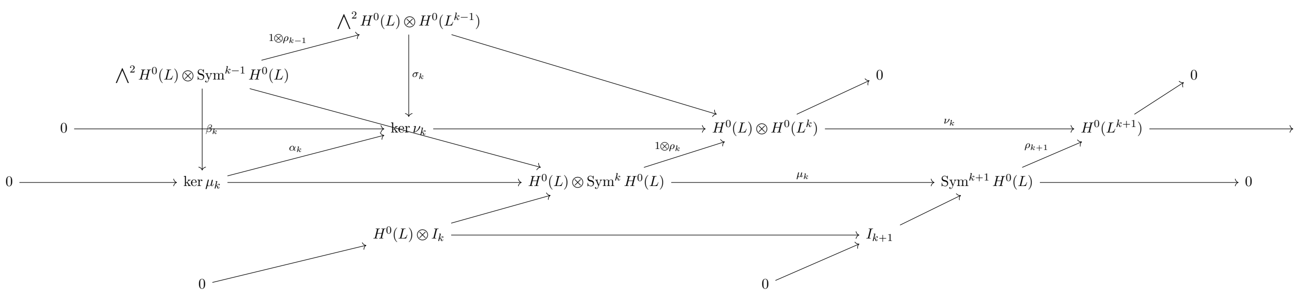
3. Koszul Cohomology Computes Syzygies

This section is devoted to the proof of a theorem that Koszul cohomology computes an upper bound for the degree of generators for the ideal of the embedded curve. A key definition is the map $\sigma_k : \bigwedge^2 H^0(L) \otimes H^0(L^{k-1}) \to H^0(M_L \otimes L^k)$ given by

$$(v_1 \wedge v_2) \otimes \alpha \mapsto v_1 \otimes v_2\alpha - v_2 \otimes v_1\alpha.$$

Theorem 3.1 (Koszul decription of syzygies)**.** [10, 1.3] *Suppose L is normally generated so φ_L is an embedding. Suppose $k_0 \in \mathbb{Z}$ is such that the maps $\sigma_k : \bigwedge^2 H^0(L) \to H^0(M_L \otimes L^k)$ are surjective for all $k \geqslant k_0$. Then every minimal generator for the canonical ideal of X in $\mathbb{P}^{g-1}$ has degree at most k_0.*

The commutativity of the diagram below is the proof by picture of the theorem.



To show that the diagram commutes is an involved argument, and relies on another visual theorem from topology.

Lemma 3.2 (snake lemma)**.** *If the following commutes*

$$\begin{array}{ccccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow b & & \downarrow c & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & & \end{array}$$

the sequence

$$\ker(a) \to \ker(b) \to \ker(c) \xrightarrow{d} \operatorname{coker}(a) \to \operatorname{coker}(b) \to \operatorname{coker}(c)$$

is exact, where d denotes a connecting homomorphism.

Proof. See page 792 in [4]. □

This one is just for intuition.

Lemma 3.3 (Symmetric-Tensor-Exterior algebra sequence)**.** *Let M be a free R-module of rank n, where R contains $\frac{1}{2}$. Then the following sequence is exact*

$$0 \to \mathrm{Sym}^2 M \to M \otimes M \to \bigwedge^2 M \to 0.$$

Proof. Let $\wedge : M^{\otimes 2} \to \bigwedge^2 M$ be the map $a \otimes b \mapsto a \wedge b$. To make this explicit, suppose $a = \sum_{i=1}^n a_i x_i$ and $b = \sum_{j=1} b_j x_j$. Then the exterior algebra relation $x \otimes x = 0$ forces $(x+y) \otimes (y+x) = 0$ and $(x \otimes y) + (y \otimes x) = 0$, which means

$$\begin{aligned} a \wedge b \ &= (\textstyle\sum_{i=1}^n a_i x_i) \wedge (\sum_{j=1}^n b_j x_j) \\ &= a \otimes b - \textstyle\sum_{i=1}^n a_i x_i \otimes b_i x_i. \end{aligned}$$

The exterior algebra $\bigwedge M$ is a well known quotient of $\oplus_{n \in \mathbb{N}} M^{\otimes n}$ and the map $\wedge$ is surjective. Let $s : \operatorname{Sym}^2 M \to M^{\otimes 2}$ be the map $m_1 m_2 \mapsto \frac{1}{2} \sum_{\sigma \in S_2} m_{\sigma(1)} \otimes m_{\sigma(2)} = \frac{1}{2} [m_1 \otimes m_2 + m_2 \otimes m_1]$. Since $ab - (-1)^{\deg a \deg b} ba = 0$ in $\operatorname{Sym}^2(M)$,

$$\begin{array}{lrl} & m_1 \otimes m_2 + m_2 \otimes m_1 & = 0 \\ \iff & m_1 m_2 & = (-1)^{\deg a \deg b} m_2 m_1 \\ \iff & m_1 m_2 & = 0 \\ \text{or} & m_1 & = m_2 \text{ with } \deg m_1 \text{ odd}, \end{array}$$

and in the latter case $m_1^2 = 0 \in \operatorname{Sym}^2 M$. So $\ker(s) = 0$ and s is injective. Then $\operatorname{im}(s) \subset \ker(\wedge)$ since

$$m_1 \wedge m_2 + m_2 \wedge m_1 = m_1 \wedge m_2 - m_1 \wedge m_2 = 0$$

and finally $\ker(\wedge) \subset \operatorname{im}(s)$ since

$$(x \otimes y) + (y \otimes x) = 2s(xy)$$

and

$$(x+y) \otimes (y+x) = s((x+y)(y+x)) - s((y+x)(x+y)).$$

□

Finally, with a restatement for convenience, the Koszul computation theorem and a proof.

Theorem 3.4. [10, 1.3] *Let X be a genus 5 non-hyperelliptic, canonical, smooth, irreducible, complex algebraic curve and let L be a line bundle on X. Suppose L is normally generated so φ_L is an embedding. Suppose $k_0 \in \mathbb{Z}$ is such that the maps $\sigma_k : \bigwedge^2 H^0(L) \to H^0(M_L \otimes L^k)$ are surjective for all $k \geqslant k_0$. Then every minimal generator for the canonical ideal of X in $\mathbb{P}^{g-1}$ has degree at most k_0.*

Proof. Every minimal generator for the canonical ideal of X in $\mathbb{P}^r$ has degree at most k_0 if and only if the maps $H^0(L) \otimes I_k \to I_{k+1}$ are surjective for all $k \geqslant k_0$. This statement means that I_{k_0} generates I as a graded ring. Let $\rho_k : \operatorname{Sym}^k H^0(L) \to H^0(L^k)$ be the surjective maps from the definition of a normally generated line bundle, Definition 2.7. Then $\ker \rho_k = I_k$ and the following commutes.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \ker(\mu_k) & \xrightarrow{\alpha_k} & \ker(\nu_k) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(L) \otimes I_k & \longrightarrow & H^0(L) \otimes \operatorname{Sym}^k(H^0(L)) & \xrightarrow{1 \otimes \rho_k} & H^0(L) \otimes H^0(L^k) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \mu_k} & & \downarrow{\scriptstyle \nu_k} & & \\
0 & \longrightarrow & I_{k+1} & \longrightarrow & \operatorname{Sym}^{k+1}(H^0(L)) & \xrightarrow{\rho_{k+1}} & H^0(L^{k+1}) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where the vertical maps are multiplication in their respective graded rings. The lower horiztonal short exact sequence is the exact sequence induced by the assumption that ρ_{k+1} is surjective and the likewise the upper short exact sequence is induced by ρ_k but with the tensor preserving exactness. It is nontrival that the tensor preserves exactness but this follows from right exactness of the ρ-sequences per [19, Tag 00CW]. By the snake lemma 3.2, $H^0(L) \otimes I_k \to I_{k+1}$ is surjective if $\alpha_k : \ker \mu_k \to \ker \nu_k$ is surjective. There are two Koszul complex with maps that takes values in $\ker \mu_k$ and $\ker \nu_k$ respectively and the normal generation of the line bundle relates these complexes so it is possible to show that α_k is surjective with a computation in Koszul cohomology.

Let $\beta_k = d_{2,k-1}^{(\mathrm{Sym}\, H^0(L))} : \bigwedge^2 H^0(L) \otimes \mathrm{Sym}^{k-1} H^0(L) \to H^0(L) \otimes \mathrm{Sym}^k H^0(L)$ be the maps

$$(v_1 \wedge v_2) \otimes \alpha \mapsto v_1 \otimes (v_2 \cdot \alpha) - v_2 \otimes (v_1 \cdot \alpha).$$

Then β_k is realized in $\ker \mu_k$ as the symmetric relation $x \otimes y - y \otimes x = 0$ forces

$$v_1 \otimes (v_2 \cdot \alpha) - v_2 \otimes (v_1 \cdot \alpha) \mapsto \alpha \otimes (v_1 \otimes v_2) - \alpha \otimes (v_2 \otimes v_1) = 0.$$

By definition 2.19 $\beta_k = (\mathrm{Id}_{\mathrm{H^0(L)}} \otimes \mu_{k-1}) \circ (\psi_{\mathrm{id}} \otimes \mathrm{Id}_{\mathrm{Sym^{k-1}\, H^0(L)}})$, ψ_{id} is dual to the exterior product which is injective, and μ_{k-1} is surjective so β_k is surjective onto $\ker \mu_k$.

Turning to $\ker \nu_k$, recall the pullback of the Euler sequence on $\mathbb{P}^r$ from Lemma 2.15. Twist the sequence by L^k and take global sections so that the following is exact

$$H^0(M_L \otimes L^k) \xrightarrow{f} H^0(L) \otimes H^0(L^k) \xrightarrow{\nu_k} H^0(L^{k+1})$$

and $\ker \nu_k = f_* H^0(M_L \otimes L^k)$. To make it more clear how Koszul cohomology will compute these global sections, the pushfoward will be abusively written as just $H^0(M_L \otimes L^k)$, but keep in mind that this is $H^0(M_L \otimes L^k) \subset H^0(L) \otimes H^0(L^k)$. The global sections of the sequence from Lemma 2.17 form the exact sequence

$$\bigwedge^2 H^0(M_L) \otimes H^0(L^{k-1}) \to \bigwedge^2 H^0(L) \otimes H^0(L^k) \xrightarrow{\sigma_k} H^0(M_L \otimes L^k)$$

where $\sigma_k = d_{2,k-1}^{(\oplus_{k \in \mathbb{N}} H^0(L^k))}$ is a Koszul map with exact the same form at β_k but the Koszul complex is with respect to a different graded algebra over $H^0(L)$. Just as with β_k, $\operatorname{im} \sigma_k \subseteq \ker \nu_k$ but this time the matter is subtler, since there is apparently no symmetric relation to fall back on. But

$$\begin{aligned} \nu_k(\sigma((s_1 \wedge s_2) \otimes f)) &= \nu_k(s_1 \otimes s_2 f - s_2 \otimes s_1 f) \\ &= s_1 s_2 f - s_2 s_1 f \\ &= 0, \end{aligned}$$

since ρ_{k-1} is surjective so s_1 and s_2 are the image of some symmetric tensors and $s_1 s_2 = s_2 s_1$. Then the following commutes

$$\begin{array}{ccc} \bigwedge^2 H^0(L) \otimes \mathrm{Sym}^{k-1} H^0(L) & \xrightarrow{\beta_k} & \ker \mu_k \\ \downarrow{\scriptstyle 1 \otimes \rho_{k-1}} & & \downarrow{\scriptstyle \alpha_k} \\ \bigwedge^2 H^0(L) \otimes H^0(L^{k-1}) & \xrightarrow{\sigma_k} & \ker \nu_k \end{array}$$

and if σ_k is surjective then so is α_k. □

4. Noether's theorem

In this section, Noether's theorem is proved as a consequence of Theorem 3.1.

Theorem 4.1 (Noether)**.** [10] *A canonically embedded nonhyperelliptic curve $X \subseteq \mathbb{P}^{g-1}$ with genus g is projectively normal. That is to say the maps*

$$H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(k)) \to H^0(X, \Omega_X^k)$$

are surjective for all $k \geqslant 0$.

One useful fact to have on hand for the proof of Noether's theorem is the example of the wedged pulled back Euler sequence 2.2. Let $M_\Omega = \varphi_\Omega^* \Omega_{\mathbb{P}^r}(1)$ and let $Q_\Omega = M_\Omega^\vee$ be the $\mathcal{O}_X$-dual. The following is exact.

$$0 \to Q_\Omega \otimes \Omega^{-l-1} \to \left(\bigwedge^2 H^0(\Omega)^\vee\right) \otimes_{\mathbb{C}} \Omega^{-l} \to \left(\bigwedge^2 Q_\Omega\right) \otimes \Omega^{-l} \to 0. \tag{4.1}$$

This series of lemmas introduces an exact sequence which is derived under the assumption that the line bundle being studied is very ample and which explains some vanishing of global sections in Lemma 4.4.

Lemma 4.2. [10] *Let X be a non-hyperelliptic genus 5 canonical, smooth, irreducible, complex algebraic curve and let $\varphi : X \to \mathbb{P}^4$ be the map obtained from global sections of the canonical bundle. Let $D = x_1 + \cdots + x_{g-2} \in \mathrm{Div}(X)$, where the x_i are points in X of general position which are distinct and linearly independent in $\mathbb{P}^{g-1}$.*
Let Λ_D be the $(g-3)$-plane in $\mathbb{P}^{g-1}$ spanned by D.
Let $L = \Omega(-D)$ and suppose that L is very ample. Then

1. *Λ_D is the subspace $\mathbb{P}(W_D) \subset \mathbb{P}(H^0(\Omega))$ where $W_D = H^0(\Omega)/H^0(\Omega(-D))$.*
2. *There is a surjection of sheaves on X, $u_D : W_D \otimes_{\mathbb{C}} \mathcal{O}_X \to \Omega \otimes \mathcal{O}_D$.*
3. *$\Lambda_D \cap X = D$ as schemes.*
4. *$h^0(\Omega(-D)) = 2$.*
5. *$M_{\Omega(-D)} = \Omega^\vee(D)$.*
6. *Let $\Sigma_D = \ker u_D$. Then $\Sigma_D \cong \oplus_{i=1}^{g-2} \mathcal{O}_X(-x_i)$.*

Proof.

1. The line bundle $L = \Omega(-D)$ is very ample if the induced map φ_L is a closed immersion. In other words L separates points and tangent vectors and hence there is a hyperplane, a global section s_i of $H^0(X, L)$ which passes through each x_i and not the others. The immersion of D then are those global sections of $H^0(\Omega)$ which correspond to hyperplanes intersecting in exactly $x_1, \cdots, x_{g-2}$, where from [25,] this the set
$$W_D = \{s \in H^0(\Omega) : \mathrm{div}\, s + D = 0\} = H^0(\Omega)/H^0(\Omega(-D)).$$
2. Recall from Lemma 2.15 the sequence
$$0 \to M_\Omega \to H^0(\Omega) \otimes_{\mathbb{C}} \mathcal{O}_X \to \Omega \to 0.$$
The map u_D corresponds to the map $H^0(\Omega) \otimes_{\mathbb{C}} \mathcal{O}_X \to \Omega$ given by the pullback by φ_L of
$$(s_0, \cdots, s_{g-2}) \mapsto x_0 s_0 + \cdots + x_{g-2} s_{g-2}$$
and therefore is given by a map of the same form. The correspondence is in the sense of the diagram [10, 2.1] abbreviated below
u_D is surjective since the Euler sequence is exact.
3. D is naturally a subscheme of $\mathbb{P}(W_D)$ so since by assumption D spans Λ_D this step follows from the dinstinctness and independence of points in general position.

$$\begin{array}{ccc} H^0(\Omega)\otimes_{\mathbb{C}}\mathcal{O}_X & \longrightarrow & \Omega \\ \downarrow & & \downarrow \\ W_D & \xrightarrow{u_D} & \Omega\otimes\mathcal{O}_D \end{array}$$

(4) By Riemann Roch since $(\deg D = g-3) < 2g-1$,

$$\begin{aligned} h^0(X,L) - h^0(X, K_X\otimes L^{-1}) &= \deg L + 1 - g \\ h^0(X,L) - (2g-2-2g-2) &= g-3+1-g \\ h^0(X,L) - 4 &= -2 \end{aligned}$$

so $h^0(\Omega(-D)) = 2$.

(5) Recall that $M_{\Omega(-D)}$ is defined by $\varphi^*_{\Omega(-D)}\Omega_{\mathbb{P}^{g-1}}(1)$. To identify this with $\Omega^\vee(D) = T_X(D)$ consider another version of Lemma 2.15

$$0 \to M_{\Omega(-D)} \to H^0(\Omega(-D))\otimes_{\mathbb{C}}\mathcal{O}_X \to \Omega(-D) \to 0$$

which is exact since $L = \Omega(-D)$ is very ample by assumption. The original version of the Euler sequence Definition 2.12 twisted by D is the exact sequence

$$0 \to \Omega_{\mathbb{P}^{g-1}}(D) \to \mathcal{O}_{\mathbb{P}^{g-1}}(D-1)^{\oplus g} \to \mathcal{O}_{\mathbb{P}^{g-1}}(D) \to 0$$

so since by the previous part of the lemma $h^0(\Omega(-D)) = 2$, taking the $\mathcal{O}_X$-duals the pullbacks of the Euler sequences must give the same exact sequences.

(6) This is a decomposition of the maps with form $(s_0,\cdots,s_{g-2}) \to x_0s_0+\cdots+x_{g-2}s_{g-2}$ into the components $s_i \mapsto x_is_i$.

□

In the context of the previous lemma there is the following useful sequence of vector bundles.

Lemma 4.3. [10, 2.3] *Let X be a non-hyperelliptic genus 5 canonical, smooth, irreducible, complex algebraic curve and let $\varphi : X \to \mathbb{P}^4$ be the map obtained from global sections of the canonical bundle. Write $\Omega = \omega_X$, let $M_\Omega = \varphi^*_\Omega\Omega_{\mathbb{P}^r}(1)$ and let $Q_\Omega = M^\vee_\Omega$ be the $\mathcal{O}_X$-dual.*

(1) *The following is exact*

$$0 \to M_{\Omega(-D)} \to M_\Omega \to \Sigma_D \to 0,$$

(2) *By Lemma 4.2 the following is exact*

$$0 \to \Omega^\vee(D) \to M_\Omega \to \oplus_{i=1}^{g-2}\mathcal{O}_X(-x_i) \to 0.$$

Proof.

(1) Recall the definitions $M_{\Omega(-D)} = \varphi^*_{\Omega(-D)}\Omega_{\mathbb{P}^{g-1}}(1)$ and $M_\Omega = \varphi^*_\Omega\Omega_{\mathbb{P}^{g-1}}(1)$. Let $i : D \hookrightarrow X$ be the inclusion of the divisor. Since D is effective and very ample by assumption and X is projective the map is a closed immersion so there is an exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to i_*\mathcal{O}_D \to 0$$

where the maps are respectively the inclusion of regular functions which vanish at $-D$ and the quotient map by that inclusion. Taking Euler sequences (vertically, on each term) gives the following exact sequence

$$0 \to \Omega_X(-D) \to \Omega_X \to \Omega_X\otimes\mathcal{O}_D \to 0.$$

This is just inclusion of holomorphic differentials with fixed zeros followed by the quotient by the inclusion. The pullbacks need to commute with this sequence which makes

$$0 \to M_{\Omega(-D)} \to M_\Omega \to \Sigma_D \to 0$$

exact, so now the inclusion is happening on the curve itself rather than in the projective space containing the embeddings.

(2) In this proof let $g = 5$ so that $D = x_1 + x_2 + x_3$ and Λ_D is the 2-plane in $\mathbb{P}^4$ spanned by these points. Consider the flag of linear spaces $\Lambda_0 \subset \Lambda_1 \subset \Lambda_D$ corresponding to the divisors $D_0 = x_1$, $D_1 = x_1 + x_2$ and D itself respectively. Let $E_0 = D_0 = x_1$, let $E_1 = x_2$ and let $E_2 = x_3$. Then there is filtration of Σ_D by vector bundles
$$\Sigma_D \supset F_1 \supset F_2 \supset 0$$
such that $F_i/F_{i+1} = \mathcal{O}_X(-E_i)$ by [20, Tag 0120].

□

This next result about global sections allows for a proof of a dual version of Noether's theorem.

Lemma 4.4 ([10] Corollary 2.4)**.**
Let X be a non-hyperelliptic genus 5 canonical, smooth, irreducible, complex algebraic curve and let $\varphi : X \to \mathbb{P}^4$ be the map obtained from global sections of the canonical bundle. Write $\Omega = \omega_X$, let $M_\Omega = \varphi_\Omega^* \Omega_{\mathbb{P}^r}(1)$ and let $Q_\Omega = M_\Omega^\vee$ be the $\mathcal{O}_X$-dual. Let $Q_\Omega = M_\Omega^\vee$. Then for each $l \geqslant 1$,

(1) $H^0(Q_\Omega \otimes \Omega^{-l}) = 0$
(2) $H^0(\bigwedge^2 Q_\Omega \otimes \Omega^{-l}) = 0$.

Proof. Consider $Q_\Omega = M_\Omega^\vee$, where
$$M_\Omega = \phi_L^* \Omega_{\mathbb{P}^r}(1) = \phi_L^* \Omega_{\mathbb{P}^r} \otimes L.$$
Taking the dual of the exact sequence Lemma 4.3, and then tensoring by Ω^{-l} gives the sequence
$$0 \to (\oplus_{i=1}^{g-2} \mathcal{O}_X(x_i)) \otimes \Omega^{-l} \to Q_\Omega \otimes \Omega^{-l} \to \Omega(-D) \otimes \Omega^{-l} \to 0.$$
The induced long exact sequence is
$$H^0((\oplus_{i=1}^{g-2} \mathcal{O}_X(x_i)) \otimes \Omega^{-l}) \to H^0(Q_\Omega \otimes \Omega^{-l}) \to H^0(\Omega_X(-D) \otimes \Omega^{-l}) \to \cdots$$
where
$$H^0((\oplus_{i=1}^{g-2} \mathcal{O}_X(x_i)) \otimes \Omega^{-l}) = H^0(\oplus_{i=1}^{g-2} \Omega^{-l}(x_i)) = \oplus_{i=1}^{g-2} H^0(-lK_X + x_i),$$
for K_X a canonical divisor, and where
$$H^0(\Omega(-D) \otimes \Omega^{-l}) = H^0(K_X - D - lK_X) = H^0(-(l-1)K_X - D).$$
Since $\deg(-lK_X + x_i) < 0$, $H^0(-lK_X + x_i) = 0$ and likewise since $\deg -(l-1)K_X - D) < 0$ for all $l \geqslant 1$, both of the H^0's surrounding $H^0(Q_\Omega \otimes \Omega^{-l})$ are 0 and $H^0(Q_\Omega \otimes \Omega^{-l}) = 0$. Taking the induced long exact sequence from (4.1),
$$H^0(Q_\Omega \otimes \Omega^{-l-1}) \to \bigwedge^2 H^0(\Omega)^\vee \otimes H^0(\Omega^{-l}) \to H^0(\bigwedge^2 Q_\Omega \otimes \Omega^{-l}) \to \cdots,$$
by the argument above
$$\bigwedge^2 H^0(\Omega)^\vee \otimes H^0(\Omega^{-l}) \cong H^0(\bigwedge^2 Q_\Omega \otimes \Omega^{-l})$$
and again by the argument above the right hand side vanishes by degree considerations. □

This next lemma is equivalent to Theorem 3.1 if Lemma 4.4 and Noether's theorem hold. It also makes for a convenient proof of Noether's theorem and is a purely cohomological version of Petri's theorem.

Lemma 4.5. [10, Corollary 1.7] *Let X be a non-hyperelliptic genus 5 canonical, smooth, irreducible, complex algebraic curve and let $\varphi : X \to \mathbb{P}^4$ be the map obtained from global sections of the canonical bundle. Write $\Omega = \omega_X$, let $M_\Omega = \varphi_\Omega^* \Omega_{\mathbb{P}^r}(1)$ and let $Q_\Omega = M_\Omega^\vee$ be the $\mathcal{O}_X$-dual. Suppose $H^0(\bigwedge^2 Q_\Omega \otimes \Omega^{-l}) = 0$ for all $l \geqslant 1$ and the map*

$$\bigwedge^2 H^0(\Omega)^\vee \to H^0(\bigwedge^2 Q_\Omega)$$

from the sequence (4.1) is surjective. Then the homogeneous ideal of X in its canonical embedding is generated by quadrics.

Proof. By Lemma 4.4 $H^0(\bigwedge^2 Q_\Omega \otimes \Omega^{-l}) = 0$ and the map $\bigwedge^2 H^0(\Omega)^\vee \to H^0(\bigwedge^2 Q_\Omega)$ is injective. Therefore it is enough to show that $\dim H^0(\bigwedge^2 Q_\Omega) = \dim \bigwedge^2 H^0(\Omega)^\vee = \binom{g}{2}$ to conclude that the map in the statement is surjective. By Noether's theorem, Ω is normally generated since it is projectively normal in its embedding and nonhyperelliptic, so the maps ρ_k from Theorem 3.1 are surjective for $k \geqslant 0$. The punchline of this lemma is a specific version of Theorem 3.1 so the game is to show the maps σ_k from 3.1 are surjective for $k \geqslant 2$. Let $l = k - 2$ and let ψ_k be the maps in the long exact sequence induced by the sequence (4.1)

$$H^0(Q_\Omega \otimes \Omega^{-l-1}) \to \bigwedge^2 H^0(\Omega)^\vee \otimes H^0(\Omega^{-l}) \xrightarrow{\psi_{l+2}} H^0(\bigwedge^2 Q_\Omega \otimes \Omega^{-l}) \to \cdots .$$

Note that ψ_k is surjective for $k \geqslant 2$ by the hypotheses, but in practice the important feature of these maps is their transpose. Recall the sequence Lemma 2.17 where wedge products of a pullback of Euler are twisted by L^{k-1}, and write down the long exact sequence

$$\cdots \to H^0(M_L \otimes L^k) \to H^1(\bigwedge^2 M_L \otimes L^{k-1}) \xrightarrow{\tau_k} \bigwedge^2 H^0(L) \otimes H^1(\Omega^{k-1}) \to \cdots .$$

By duality τ_k is the transpose ψ_k^T, so since ψ_k is surjective, τ_k is injective, but τ_k is injective if and only if σ_k is surjective. By Theorem 3.1 the homogeneous ideal of X in its embedding is generated by quadrics. $H^0(\bigwedge^2 Q_\Omega \otimes \Omega^{-l}) = 0$, □

Finally, with the tools used to prove Lemma 4.5 in mind, Noether's theorem can be proved.

Proof. This is a proof of Theorem 4.1.
Recall that Ω is normally generated if and only if $H^1(M_\Omega \otimes \Omega^{-k}) \to H^0(\Omega) \otimes H^1(\Omega^k)$, from the twist of the pulled back Euler sequence Lemma 2.15, are injective by Lemma 4.4. But given Lemma 4.5, those maps are injective if and only if the injective maps $H^0(\Omega)^\vee \to H^0(Q_\Omega)$ are surjective. Recall that Ω is very ample by assumption, and the following sequence, a filtration of M_Ω, is exact

$$0 \to \Omega^\vee(D) \to M_\Omega \to \oplus_{i=1}^{g-2} \mathcal{O}_X(-x_i) \to 0.$$

Therefore

$$\begin{aligned} h^0(Q_\Omega) &\leqslant h^0(\Omega(-D)) &+ \textstyle\sum_{i=1}^{g-2} h^0(\mathcal{O}_X(x_i)) \\ &= 2 &+(g-2) \\ &= h^0(\Omega). \end{aligned}$$

□

5. Koszul Cohomological Proof of Petri's Theorem

In this section, the Koszul cohomology with particular assistance of the previous two sections' main results, will be used to prove Petri's theorem for the case of a genus 5, non-exceptional curve.

Let X be a non-hyperelliptic, smooth, irreducible, projective complex curve of genus 5. To prove Petri's result that $I_{X/\mathbb{P}^4}$ is generated by quadrics, [10] use Lemma 4.5 and the essence of the argument is to demonstrate that

$$h^0(\bigwedge^2 Q_\Omega) \leqslant \binom{5}{2}.$$

The proof of that fact is the main purpose of this section because of Lemma 4.4, restated here for convenience.

Fact 5.1. *$H^0(\bigwedge^2 Q_\Omega \otimes \Omega^{-l}) = 0$ for all $l \geqslant 1$.*

Proof. This is part of Lemma 4.4. □

Fact 5.2. *The map $\bigwedge^2 H^0(\Omega)^\vee \to H^0(\bigwedge^2 Q_\Omega)$ induced from Lemma 4.1 is injective.*

Proof. This is the other part of Lemma 4.4. □

Finally for clarity a version of the uniform position theorem of [2] is stated in the language of [10].

Lemma 5.3. *GL An effective divisor E of degree k spans a $(k-r-1)$-plane in $\mathbb{P}^{g-1}$ if and only if it moves in a linear system of dimension r.*

Now the proof of Petri's theorem can proceed as in [10].

Theorem 5.4. [10] *Let X be a non-hyperelliptic, smooth, irreducible, projective complex curve of genus 5. Suppose A is a degree 4 line bundle on X with $h^0(A) = 2$ such that A and $\omega \otimes A^\vee$ are generated by global sections. The the homogeneous ideal of X in its canonical embedding $I_{X/\mathbb{P}^4}$ is generated by forms of degree 2.*

Before the proof, one more sequence must be exact, but the proof of exactness is nontrivial and the setup is based on the statement of Petri's theorem, so it will be stated as a result in its own right here.

Let $A \in W^1_4(X)$ be the line bundle from the statement of Theorem 5.4. Let $D = (\operatorname{div} f)$ for some $f \in H^0(X, A)$. Since A is generated by global sections and all of the spaces in consideration lie over $\mathbb{C}$ which has characteristic 0

$$D = x_1 + \cdots + x_4,$$

for some distinct x_i.

Corollary 5.5. [10] *Let $D = x_1 + \cdots + x_4$ be as above for some distinct x_i. No effective divisor contained in D can move in a nontrivial linear series.*

Proof. Suppose such a divisor existed. The $|D|$ either has a base point or dimension at least 2 both of which contradict global generation and uniform position per Lemma 5.3. □

In $\mathbb{P}^4 = \operatorname{Proj}(H^0(X, \omega))$, D spans a 2-plane Λ_D and by Lemma 5.3 any proper subset of the x_i are linearly independent.

Corollary 5.6. [10] *Let X be a non-hyperelliptic genus 5 canonical, smooth, irreducible, complex algebraic curve and let $\varphi : X \to \mathbb{P}^4$ be the map obtained from global sections of the canonical bundle. Let $D = x_1 + \cdots + x_4$ for some distinct closed points x_i in general position. Write $\Omega = \omega_X$, let $M_\Omega = \varphi^*_\Omega \Omega_{\mathbb{P}^r}(1)$ and let $Q_\Omega = M^\vee_\Omega$ be the $\mathcal{O}_X$-dual. Let $M_{\Omega(-D)} = \Omega^\vee(D)$. Then the following is exact*

$$0 \to M_{\Omega(-D)} \to M_\Omega \to \Sigma_D \to 0.$$

Proof. Since $\omega \otimes A^\vee = \Omega(-D)$ is generated by global sections and $h^0(\Omega(-D)) = 2$ exactness follows. □

Finally, the last new exact sequence needed to prove Petri's theorem

Lemma 5.7. [10] *Let X be a non-hyperelliptic genus 5 canonical, smooth, irreducible, complex algebraic curve and let $\varphi : X \to \mathbb{P}^4$ be the map obtained from global sections of the canonical bundle. Let $D = x_1 + \cdots + x_4$ for some distinct closed points x_i in general position. Write $\Omega = \omega_X$, let $M_\Omega = \varphi_\Omega^* \Omega_{\mathbb{P}^r}(1)$ and let $Q_\Omega = M_\Omega^\vee$ be the $\mathcal{O}_X$-dual. The sequence*

$$0 \to \mathcal{O}_X(-x_{g-2} - x_{g-1}) \to \Sigma_D \to \oplus_{i=1}^{g-3} \mathcal{O}_X(-x_i) \to 0$$

is exact.

Proof. Let $D' = x_1 + x_2$ and let $E = x_3 + x_4$. Then $\Omega(-D')$ is generated by global sections since the only possible base points are x_3 and x_4 but if either were a base point then some $(g-2)$ of the $\{x_i\}$ would lie in the $(g-4)$-plane $\Lambda_{D'}$ spanned by D'. Let $V = H^0(\Omega(-D'))/H^0(\Omega(-D))$ and let $\tilde{v_E} : H^0(\Omega(-D')) \otimes_{\mathbb{C}} \mathcal{O}_X \to \Omega \otimes \mathcal{O}_E$ be the natural map defined by evaluating sections of $\Omega(-D')$ on E. Let $v_E : V \otimes_{\mathbb{C}} \mathcal{O}_X \to \Omega \otimes \mathcal{O}_E$ be the induced map. As effective divisors D and D' span hyperplanes λ_D and $\lambda_{D'} \subset \mathbb{P}^4$ which in particular are the subspaces

$$\Lambda_D = \mathbb{P}(W_D),\ \Lambda_{D'} = \mathbb{P}(W_{D'}) \subset \mathbb{P}(H^0(\Omega)).$$

Then the following commutes.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & V \otimes_{\mathbb{C}} \mathcal{O}_X & \xrightarrow{v_E} & \Omega \otimes \mathcal{O}_E & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Sigma_D & \longrightarrow & W_D \otimes_{\mathbb{C}} \mathcal{O}_X & \xrightarrow{u_D} & \Omega \otimes \mathcal{O}_D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Sigma_{D'} & \longrightarrow & W_{D'} \otimes_{\mathbb{C}} \mathcal{O}_X & \longrightarrow & \Omega \otimes \mathcal{O}_{D'} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

If $s \in H^0(\Omega(-D'))$ is some section which does not vanish on D then s cannot vanish at x_3 or x_4. So $\tilde{v_E}$ and therefore v_E are surjective. Since $\dim_{\mathbb{C}} V = 1$ implies that $\ker v_E \cong \mathcal{O}_X(-E)$ the following is exact

$$0 \to \mathcal{O}_X(-x_3 - x_4) \to \Sigma_D \to \Sigma_{D'} \to 0.$$

Finally since D' is composed of a pair of linearly independent points spanning a line $\Lambda_{D'}$

$$\Sigma_{D'} = \mathcal{O}_X(-x_1) \oplus \mathcal{O}_X(-x_2).$$

□

Recall that the goal is to prove Theorem 5.4.

Proof. By the exactness of Corollary 5.6 the following is exact

$$0 \to \bigwedge^2 (\Sigma_D^\vee) \to \bigwedge^2 Q_\Omega \to \Sigma_D^\vee \otimes \Omega(-D) \to 0.$$

The exactness of Lemma 5.7 implies that

$$0 \to \bigwedge^2 (\mathcal{O}_X(x_1) \oplus \mathcal{O}_X(x_2)) \to \bigwedge^2 \Sigma_D^\vee \to \mathcal{O}_X(x_1 + x_3 + x_4) \oplus \mathcal{O}_X(x_2 + x_3 + x_4) \to 0$$

and

$$0 \to \Omega(-D + x_1) \oplus \Omega(-D + x_2) \to \Sigma_D^\vee \otimes \Omega(-D) \to \Omega(-D + x_3 + x_4) \to 0$$

are exact. Finally since $g = 5$ all of the divisors in two previous exact sequences above are properly contained in D so each has a unique section and

$$h^0(\bigwedge^2 \Sigma_D^\vee) \leqslant \binom{2}{2} + (g - 3).$$

Then since $h^0(\Omega(-D + x_i)) = 2$ for each i but $h^0(\Omega(-D + x_3 + x_4)) = h^0(\Omega(-D')) = 3$ it follows that

$$h^0(\Sigma_D^\vee \otimes \Omega(-D)) \leqslant 2(g - 3) + 3.$$

By the exactness of

$$0 \to \bigwedge^2 \Sigma_D^\vee \to \bigwedge^2 Q_\Omega \to \Sigma_D^\vee \otimes \Omega(-D) \to 0,$$

conclude that

$$h^0(\bigwedge^2 Q_\Omega) \leqslant \binom{2}{2} + 3(g - 3) + 3 = \binom{5}{2}.$$

□

6. Literature Review and Statement of Thesis-Like Problems

To conclude the document, some motivations for doing this problem are stated. The idea behind the example which this work explains is to re-derive the Petri equations for a canonical curve. Eventually the goals will be to write equations for canonical varieties, surfaces in particular, and for stacks. An example of a problem which this theory might address is to write down the algebra of modular forms for a congruence subgroup in the function field setting. This section verifies that Koszul cohomology computations apply to varieties of higher dimension than curves, introduces canonical surfaces with example of surfaces for which equations are known and demonstrates some known results about the section rings of stacks. It turns out that Koszul cohomology is related to log stacky curves by the $K_{p,1}$-theorem, which states that the order p relations, or relations of relations of relations so on p times, among degree $p + 1$ generators for the embedded curve, both canonical and stacky, can be computed with a Koszul cohomology group. As a subject, this might be called studying Torelli problems or the recovery of suitably nice varieties from abstract principally polarized abelian varieties such as the Jacobian of a smooth genus g curve.

6.1. $K_{p,1}$ and other Syzygy Theory.

In [11] Green makes good on a claim in [10] that the statement and proof of Theorem 3.1 can be done for any projective variety over an algebraically closed base field with arbitrary characteristic. A key technique is the relation of exact sequences of the pulled back bundle M_Ω with line bundle quotients corresponding to secant planes to the canonical curve. The embedding need not even be canonical, as any very ample line bundle has a flag of linear spaces which gives the same sort of filtration. Then the $K_{p,1}$ theorem is a relation between syzygies in the ideal of a Cox ring and the E_1 page of a spectral sequence in a way that gives Green fairly immediate generalizations to projective varieties. This section makes these claims more precise and includes a conjecture about surfaces.

Lemma 6.1. [11, 0.17] *Let X be a projective variety of dimension at least 2 with $K_X \leqslant 0$, and let L is an ample line bundle on X. Then if $X \cap H$ is a smooth hyperplane section it is connected and $H^1(X, qL) = 0$ for all $q \geqslant 0$.*

In this situation Green is able to conclude a 'Lefschetz' theorem.

Theorem 6.2. [11, 3.b.7] *With the hypotheses of Lemma 6.1, $K_{p,q}(X, L) \cong K_{p,q}(X \cap H, L)$.*

Example. Theorem 6.2 holds for $K3$ surfaces and Fano 3-folds.

Theorem 6.3. [11, 3.c.1] *Let $m = \dim \phi_L(X)$ and suppose $h^0(X, L) = r + 1$. Then*

$$\begin{cases} K_{p,1}(X, L) = 0, & \textit{for } p > r - m \\ K_{r-m,1}(X, L) = 0, & \textit{unless } \phi_L(X) \textit{ is an } m\textit{-fold of minimal degree} \\ K_{r-m-1,1}(X, L) = 0, & \textit{unless } \deg \phi_L(X) \leqslant r + 2 - m \textit{ or } \phi_L(X) \textit{ lies on an } (m+1)\textit{-fold of minimal degree.} \end{cases}$$

The $K_{p,1}$ theorem is the Koszul complex generalization of Petri's equations and computes those equations with Theorem **??**. The proof of Theorem **??** itself involves the promised spectral sequences and as is those sequences will eventually be useful for computing Koszul groups of surfaces they are introduced here. Fix $d_0 \in \mathbb{Z}$ and consider the bigraded complex

$$A^{-p,-q} = \begin{cases} \bigwedge^p V \otimes \oplus_{i \geqslant 0} \left(\mathrm{Sym}^i(V) \otimes M_{q,d-p-i} \right), & q \geqslant 0 \\ \bigwedge^p V \otimes B_{d-p}, & q = -1 \\ 0, & q < -1 \end{cases}$$

with maps

$$A^{p,q} \xrightarrow{d} A^{p+1,q}, \text{ where } d : \begin{cases} \bigwedge^p V \otimes \mathrm{Sym}^{l-2} V \to V \otimes \mathrm{Sym}^{l-1} V \to \mathrm{Sym}^l V, & q \geqslant 0 \\ \text{is the map from Definition 2.19}, & q = -1 \end{cases}$$

and

$$A^{p,q} \xrightarrow{\delta} A^{p,q+1}$$

which is $\bigwedge^{-p} V$ tensored with $(-1)^p$ times the degree $(d_0 - p)$ terms of the minimal free resolution

$$\cdots \to \bigoplus_{q \geqslant q_1} \mathrm{Sym}(V)(-q) \otimes M_{1,q} \to \bigoplus_{q > q_1} \mathrm{Sym}(V)(-q) \otimes M_{0,q} \to B \to 0.$$

Since $d^2 = 0$, $\delta^2 = 0$ and $d\delta + \delta d = 0$ there are two spectral sequences E and E' abutting the total complex with $E_1^{p,q} = 0$ for all p, q and

$$E_1'^{p,q} = \begin{cases} K_{-p,d_0-p}(B, V), & q = -1, \text{ and any } p \\ M_{-q,d_0}(B, V), & q \geqslant 0, \; p = 0 \\ 0, & o/w \end{cases}$$

in [11, 1.b.9].

6.2. **Surfaces.**

One extension of this kind of explicit canonical modeling is to ask for models of varieties in higher dimensions. In the spirit of induction, two dimensional varieties, or surfaces are next. Two central questions about surfaces are

(1) which varieties are canonical?
(2) given a canonical variety X, is it known how to write down its equations?

First of all a precise definition for a surface is required.

Definition 6.4. [17, 1.1.1] *A surface S is a complex, projective surface which is an irreducible and reduced algberaic variety of dimension 2 over $\mathbb{C}$.*

Recall that a variety Y is canonical if $Y \cong \mathrm{Proj}(Y, K_Y)$. Then if Y is canonical, the map $\mathbb{C}[X_0, \cdots, X_r] = R(\mathbb{P}^r, \mathcal{O}(1)) \to R(Y, K_Y)$ is surjective and it follows that the canonical ring of Y is finitely generated. A crucial part of what are secretly syzygy results in this area of math is the projective normality of the embedded object: the curves as in the Petri equation example which is most of these notes, or in this section some kind of surface. This is just by way of reminder that this discussion of normality has to do both with the bundles on the varieties in question, and the normality of the embedded object with the maps induced by those bundles per Section 2.

6.2.1. *Canonical Models and Canonical Singularities.*
Please keep in mind through this section, which defines a canonical surface, that the upshot is that the canonical model is a projective variety. That fact will become indispensible for the proof of finite generation of the canonical ring. With a few exercises from Hartshorne there is an abstract characterization of the canonical models in terms of resolutions (proper birational morphisms with a smooth variety).

The sort of catch phrase definition of the canonical bundle is the "top exterior power of the sheaf of differentials" and that statement is precise in the follow sense.

Definition 6.5. [13, 1.31] *Let X be a smooth variety over a field $\mathbb{K}$.*

(1) *the canonical sheaf of X is $\omega_X = \bigwedge^{\dim X} \Omega_{X/\mathbb{K}}$*
(2) *any divisor D such that $\mathcal{O}_X(D) \cong \omega_X$ is a canonical divisor.*

One convenient argument for smoothness of a variety is based on the composition of the variety.

Fact 6.6. *Let X be a variety over the field $\mathbb{K}$. If $\mathbb{K} = \overline{\mathbb{K}}$ then nonsingularity of X implies smoothness.*

Proof. Thanks to [25] this proof is almost immediate as the so called Jacobian criterion for smoothness immediately forces smoothness. Suppose X is a nonsingular variety of dimension r over $\mathbb{K}$ some algebraically closed field. Then $X = \mathrm{Spec}\,\mathbb{K}[x_1 \cdots, x_n]/(f_1, \cdots, f_r) \to \mathrm{Spec}\,\mathbb{K}$ is smooth of relative dimension n if it is flat of relative dimension n and the corank of the Jacobian is n. But $X \to \mathrm{Spec}\,\mathbb{K}$ is smooth if and only if X is a disjoint union of nonsingular $\mathbb{K}$-varieties of dimension n which X is by assumption. □

The notion of smoothness offers a nice characterization of the canonical class for a variety.

Definition 6.7. [13] *Let X be a normal variety over a perfect field $\mathbb{K}$. Let $j : X^{sm} \to X$ be the inclusion of the locus of smooth points. The unique linear equivalence class K_X of Weil Divisors on X such that $K_X|_{X^{sm}} = K_{X^{sm}}$ called the canonical class of X.*

The inclusion of smooth locus also gives a different expression of the canonical sheaf.

Definition 6.8. [13] *Let X be a normal variety over a perfect field $\mathbb{K}$. Let $j : X^{sm} \to X$ be the inclusion of the locus of smooth points. The pushforward $\omega_X = j_* \omega_{X^{sm}}$ is the canonical sheaf on X.*

The pushforward definition 6.8 comes with these results.

Fact 6.9.

(1) *The canonical sheaf of X is a rank 1 coherent sheaf on X.*
(2) *If X is proper then the canonical sheaf ω_X agrees with the dualizing sheaf ω_X^0.*

Finally there are enough criteria to define a canonical model.

Theorem 6.10. [13, 1.32] *A normal proper variety Y is a canonical model for X if and only if there is $m_0 > 0$ such that $m_0 K_Y$ is Cartier and ample and there is a resolution $f : X \to Y$ and an effective f-exceptional divisor E ($f(E)$ has codimension at least 2 as a subvariety of Y) such that*

$$m_0 K_Y \sim f^*(m_0 K_Y) + E.$$

Local singularities may appear on canonical models.

Definition 6.11. [13, 1.33] *A normal variety Y has canonical singularities if both*

(1) *$m_0 K_Y$ is Cartier for some $m_0 > 0$ and*
(2) *there is a resolution $f : X \to Y$ and an effective f-exceptional divisor E such that*

$$m_0 K_Y \sim f^*(m_0 K_Y) + E.$$

Here are some remarks which have nontrivial proofs about this defininition.

Fact 6.12.

(1) *This definition is independent of the choice of resolution.*
(2) *Equivalently, Y has canonical singularities if and only if every point $y \in Y$ has an etale neighborhood which is an open subset on some canonical model.*

Proof. From [14, 2.12] if $\tilde{f} : X \to Y$ is some other resolution than f, then write

$$K_Y \sim_{\mathbb{Q}} \tilde{f}^* K_X + \sum a_i E_i,$$

where the sum runs over $\tilde{f}$-exceptional divisors E_i. If X is canonical then $a_i \geqslant 0$ for each i and taking $m_0 = 1$ and $E = \sum a_i E_i$ makes K_Y equivalent to the resolution by f. The equivalent formulation in terms of etale maps requires a whole different resolution-free definition of canonical singularities and since those are things to be avoided for the purposes of this discussion, that proof is not included. $\square$

A complete list of canonical singularities is known in dimension 2 and a lot is known in dimension 3.

Example. [13, 1.33.3-1.33.6] Some known canonical singularities are

(1) Smooth points are canonical.
(2) The hypersurface singularity $(x_1 x_2 + f(x_3, \cdots, x_n) = 0)$ is canonical if and only if f is not indentically 0.
(3) The quotient singularity $\mathbb{A}^d / \frac{1}{n}(1, n-1, a_3, \cdots, a_d)$ is canonical for each $d \geqslant 3$ if $(n, a_i) = 1$. Its canonical class is Cartier if and only if $n \mid a_3 + \cdots + a_d$.
(4) There is a Reid-Tai criterion for canonicity of arbitrary singularities but it is not easy to write the closed form.
(5) The cone $C_d(\mathbb{P}^n)$ over the Veronese embedding $\mathbb{P}^n \to \mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}(d)))$ has a canonical singularity if and only if $d \leqslant n+1$. Its canonical class is Cartier if and only if $d \mid n+1$.
(6) General cones are covered in other works of Kollar cited in [13, 1.33.6].

6.2.2. *Classification of Surfaces.*
The classification of surfaces separates out families of canonical and other surfaces each of which is thoroughly studied in its own right. To motivate this, the classification of algebraic curves provides a convenient and historical context. A map from a nonsingular projective curve C of genus g into projective space using a multiple nK_C of the canonical class K_C, is an embedding for some $n \geqslant 3$ if $g > 1$. In particular if C is nonhyperelliptic, then $n \geqslant 1$ works, as in these notes, $n \geqslant 2$ works for hyperelliptic curves when $g > 2$ and $n \geqslant 3$ works for curves of genus 2.

Likewise with curves, a certain kind of surface, in particular surfaces of general type, have a theory of when a multiple of their canonical class gives a birational map between the surface and its embedding in projective space. There is a coarse and quasi-projective moduli space of surfaces of general type with a correspondence between pairs of Chern numbers and surfaces of general type but the description of the moduli space itself is a difficult problem. Even though in some sense most surfaces have general type, only components of the moduli space are known, and only finitely many examples. To discuss the modeling which is more the spirit of these notes it will be enough for now to define Chern classes and discuss their geography.

One intermediate definition which needs to be stated is that of the first Chern class for a surface S, denoted $c_1(S)$.

Definition 6.13 (1.3.1)**.** [**?**] *Let X be a smooth projective variety of dimension n and let E be a vector bundle of rank r on X generated by its global sections. Then $c_1(E) = c_1(\bigwedge^r E)$ is the vanishing locus of a global section of $\bigwedge^r E$.*

A fact which makes the Chern class both more intuitive and useful for a smooth projective variety at least, comes from a very involved discussion of duals, Grassmanians, ect... but is summarized here.

Fact 6.14.

(1) $c(S) = c(T_S) = 1 + c_1(T_S)t + c_2(T_S)t^2 \in A^0(S) \oplus A^1(S)t \oplus A^2(S)t^2$, *where T_S denotes the tangent bundle on S,*
(2) $c_1(E) = c_1(\det E)$ *for E a vector bundle, where* $\det E = \bigwedge^{\dim(X)} E$,
(3) $c_1(S) = c_1(\omega_S^\vee) = c_1(\det T_S) = -K_S$ *for K_S the canonical class on S.*

Here is a first definition of a surface of general type, stated in the most intuitive terms possible.

Definition 6.15. [23, 20.12] *An algebraic surface S is of general type if $P_2(S) \geqslant 1$ and $c_1^2 > 0$, where $P_2(S) = \dim_{\mathbb{C}} H^0(S, \mathcal{O}_S(2K_S))$ is the 2-plurigenus of S and c_1^2 is the self intersection multiplicity defined by $c_1^2(S) = K_S \cdot K_S$.*

Another formulation of general type now with respect to an object embedded in some space is due to the Kodaira dimension.

Definition 6.16. [23, 20.6.7; 11.1]

(1) *A complex variety V is of hyperbolic type if $\kappa(V) = \dim V$, where $\kappa(V)$ is the Kodaira dimension.*
(2) *An algebraic surface of hyperbolic type is called an algebraic surface of general type.*

The reason surfaces of general type are so distinguished here is the next theorem stated in the general form it was first proved and then restated with the particular application to surfaces of general type.

Theorem 6.17. [17, 3.3.2] *Let X be a canonical surface. If $m \geqslant 5$ then mK_X is very ample.*

Theorem 6.18. [6] *A surface S of general type has an embedding into projective space by $5K_S$ such that the image of S in its embedding is birationally equivalent to S.*

One such embedding is the following example.

Example. [6]
If $m > 4$ then a degree m nonsingular surface in $\mathbb{P}^3$ is a surface of general type and K_X itself gives an embedding.

The classification of surfaces requires one last definition.

Definition 6.19. [23, 20.2] *A surface S is called a minimal model if any bimeromorphic map $f : \tilde{S} \to S$ of a surface $\tilde{S}$ onto S is morphism.*

Finally classes of surfaces besides those of general type are given in the next theorem.

Theorem 6.20. *Surfaces S for which no mulitple nK_S gives an embedding are divided into five classes*

(1) *rational surfaces - per remark* [23, 20.17] *any rational surface free from exceptional curves of the first kind is either $\mathbb{P}^2$ or a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.*
(2) *$K3$ surfaces - definition* [23, 20.6.3] *says S is a $K3$ surface if $q(S) = 0$ and S is analytically trivial, where $q(S) = \dim_{\mathbb{C}} H^1(S^*, \mathcal{O}_{S*})$ is the irregularity of S, and S^* is a nonsingular model of S.*
(3) *ruled surfaces - definition* [23, 20.6.1] *says a surface S is a ruled surface of genus g if S is birationally equivalent to a product of $\mathbb{P}^1$ and a nonsingular genus g curve C.*
(4) *Abelian varieties - in* [21, Tag 0BF9]*, an abelian variety is a group scheme over a field $\mathbb{F}$ which is also a proper, geometrically integral variety over $\mathbb{F}$.*
(5) *surfaces with a pencil of elliptic curves - per definition* [23, 20.6.2] *a surface S is an elliptic surface if there is a surjective morphism $f : S \to C$ onto C a nonsingular curve such that the general fibre of is an elliptic curve.*

The rest of this section, following [17] is a more precise study of surfaces of general type which will include some invariant theory, Riemann-Roch for surfaces, and Enriques-Kodaira classification for surfaces of general type by those invariants, otherwise known as the "Geography of Chern numbers." After all of that, finally some motivation for this amount of attention these surfaces get is stated in some examples of interesting surfaces of general type and some theorems which connect this theory to the rest of the document.

Surfaces have four important invariants, which have not been defined before since their computation relies on some extra assumptions as in the Riemann-Roch statement after this.

Definition 6.21. [17] *Let S be a smooth surfaces.*

(1) *Geometric genus $p_g(S) = h^0(\mathcal{O}_S(K_S))$*
(2) *mth plurigenus $P_m = h^0(\mathcal{O}_S(mK_s))$*
(3) *irregularity $q = h^1(\mathcal{O}_S) = h^0(\Omega^1_S)$ per Hodge theory*
(4) *Euler characteristic $\chi = \chi(\mathcal{O}_S) = 1 - q + p_g$.*

The Euler characteristic for a smooth surface of general type is computable with the Riemann-Roch theorem for surfaces stated here.

Theorem 6.22. [17, 1.1.3] *Let S be a smooth surface and $D \in \mathrm{Div}(S)$. Then*

$$\chi(\mathcal{O}_S(D)) = \chi(\mathcal{O}_S) + \frac{D(D - K_S)}{2}.$$

With these tools, some "cartography" for the classification of surfaces becomes possible, which in particular motivates the claim that "most surfaces are of general type," and what follows in a concise summary of work by Enriques and Kodaira in classifying surfaces. The picture summary for Enriques-Kodaira classification: "Geography of Surfaces of General Type."

Now that there is some appropriate setting for them here are some examples of surfaces of general type organized by certain characteristic and nice properties.

Example.

(1) Horikawa surfaces - lie on or near the Noether line
(2) $c_2 = 3$ - so called "fake projective line"/Mumford surfaces
(3) $c_2 = 10$ - Catanese surfaces which are simply connected
(4) $c_2 = 11$ - quotient surfaces called "Godeaux surfaces"
(5) $c_2 = 11$ - arithmetic genus 0 simply connected surfaces called "Barlow surfaces"
(6) Todorov surfaces - a counterexample to the Torelli theorem

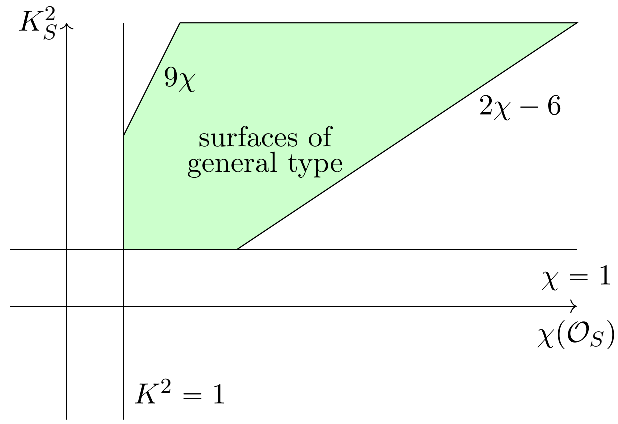


FIGURE 1. [17, Figure 2.1]

Another class of examples come from the following observations.

Example.

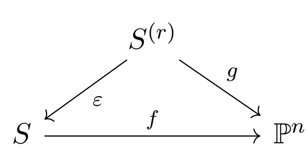
(1) if C_1, C_2 and are genus $g_1, g_2 \geqslant 2$ curves then $C_1 \times C_2$ is a minimal surface of general type with $q = g_1 + g_2,\ p_g = g_1 g_2,\ K^2 = 4(g_1 - 1)(g_2 - 1)$
(2) complete intersections of $(n-2)$ hypersurfaces in $\mathbb{P}^n$ are <u>almost</u> always minimal surfaces of general type

There is definitely immediate work to be done following up on these examples, especially in light of the following two theorems which connect this theory to the rest of these notes. First is a theorem of Castelnuovo which describes when a surface is minimal, in the sense of not admitting $E^2 = -1$ curves.

Theorem 6.23. [17, 1.2.4] *Let S' be a smooth surface and E a smooth rational curve on S' such that $E^2 = -1$. Then there exists a smooth surface S and a morphism $\phi : S' \to S$ such that π contracts E to p some point and (S', π) is isomorphic to the blow-up of S at p.*

Turning the discussion back to the filtrations or resolutions involved in4 which are more fleshed out in 6.1, there is a resolution theorem for surfaces.

Theorem 6.24. [17, 1.2.3] *Let S be a smooth surface and let $f : S \to \mathbb{P}^n$ be a rational map. Then there is a finite sequence of blow-ups of S, $\varepsilon : S^{(r)} \to S^{(r-1)} \cdots \to S^{(1)} \to S$ and a morphism $g : S^{(r)} \to \mathbb{P}^n$, a (minimal if r is) resolution of the indeterminacy locus of f, such that the following commutes.*



6.2.3. *Finite Generation of the Canonical Ring and Projective Normality for Surfaces.*
The canonical ring of a variety is known to be finitely generated in characteristic 0 but this result is nontrivial. With the intention of using as little as possible but as much as is necessary to describe what [8] says about finite generation, there are some big ideas which characterize the proof and demonstrate the features of a canonical example of surfaces for which Petri equations exist. As this result is intrinsically related to the structure of the surface, such a discussion narrows down a

good class of surfaces to follow up on.

A classical result of Zariski is that $h^0(X, \mathcal{O}_X(mK_X))$ is a bounded periodic polynomial f of degree $m \leqslant 2$ called the Kodaira dimension of the embedded variety. The Kodaira dimension 0 case is well known. In particular $\omega_S \cong \mathcal{O}_S$ and the function field $\kappa(S) = 0$ so S is either $K3$, an abelian surface, or an Enriques surface. The canonical morphism is trivial, but the arithmetic of each of these classes is complicated. Abelian surfaces come up later.

On the other hand the proofs of finite generation in the Kodaira dimensions 1 and 2 cases do respectively include the so called kitchen sink approach, or using a little bit of everything. In Kodaira dimension 1 there is some map g from the surface S to the curve C whose general fibre is an elliptic curve and an effective divisor Δ with rational coefficients which measures how far S is from being a product of C and the elliptic curve. Kodaira shows that $K_S = g^*(K_C + \Delta)$ and it follows that

$$R(S, K_S) = R(C, K_C + \Delta) = \bigoplus_{n \geqslant 0} H^0(C, \mathcal{O}_C(\lfloor m(K_C + \Delta) \rfloor))$$

which is known to be finitely generated. The subtler Kodaira dimension 2 case relies on the projectivity of the variety which allows an expression of the canonical ring as a coordinate ring under some embedding.

Theorem 6.25. [8, Base point free] *Let X be a smooth projective variety. If K_X is nef ($K_X \cdot X \geqslant 0$) and big (Kodaira dimension of $K_X = \dim_k X$) then K_X is semiample and it follows that $R(X, K_X)$ is finitely generated.*

If there is a map $f : X \to \mathbb{P}^r$ such that $D = f^*H$, for H some hyperplane, then say that D is semiample. In this case $R(X, D)$ is finitely generated. If $D = f^*H$ is semiample then it is nef: $D \cdot C \geqslant 0$ for $C \subset \mathbb{P}^r$ some curve. Therefore the problem of finite generation, with the help of a theorem of Fujino and Mori which allows the assumption that K_X or $K_X + \Delta$ is big, is all about finding nef K_X.

Theorem 6.26. [9, 37] *Let S be a minimal surface of general type. Then for $m \geqslant 4$, mK_S is base point free and if $m \geqslant 5$ the embedding morphism induced by the bundle mK_S is birational.*

Finally, some characetrization of projective normality for surfaces connects the definitions of canonical varieties, finite generation of the canonical ring and the equations for the embedded surface.
Let S be a projective surface, so a smooth projective scheme of dimension 2 over $\mathbb{C}$.
Let L be a line bundle on S which is nef and big and such that $\mathbf{L} = L + K_S$ is spanned by global sections and big.
Let $H := \mathbf{L}^{\otimes n}$ for $n \geqslant 1$ and call the associated map $\varphi_{n\mathbf{L}} : S \to \mathbb{P}^r$.

Definition 6.27. [1, 1.4] *H is projectively normal if for every $\rho \geqslant 0$ the maps*

$$H^0(\mathbb{P}^r, \mathcal{O}(\rho)) \cong \operatorname{Sym}^\rho H^0(S, H) \to H^0(S, H^\rho)$$

are surjective.

Remarkably when $\rho = 0$ the maps are surjective for all n and if $\rho = 1$ the maps surject if H is very ample. With some Koszul cohomology Andreatta and Ballico demonstrate some very clear extensions of the example in this paper.

Theorem 6.28. [1, 2.4] *Let S and L as above, and such that $h^0(L) \geqslant 4$. For each $n \geqslant 2$ if $\boldsymbol{L}^{\otimes n}$ is very ample then it is projectively normal. If $h^0(\boldsymbol{L}) \geqslant 4$ then the ideal of $\varphi_{n\boldsymbol{L}}(S)$ is generated by quadrics.*

Turning to abelian surfaces, there is a complete study of the projective normality of abelian surfaces embedded by complete linear systems.
Let A be an abelian surface.
Let L be an ample line bundle of type (n_1, n_2) on A, and denote the induced map $\varphi_L : A \to \mathbb{P}^{n_1 n_2 - 1}$.

Theorem 6.29. [7, 1.1 and summary]

(1) *If $n_1 \geqslant 3$ then φ_L is a projectively normal embedding.*
(2) *If $n_1 = 2$ then φ_L is a projectively normal embedding if and only if no point of $k(L)$ is a base point of L', where $L = L'^2$.*
(3) *If $n_1 = 1$ then L is a primitive bundle of type $(1, n_2)$ such that*
  (a) *If $n_2 = 7, 9, 11$ or $n_2 \geqslant 13$ then φ_L is projectively normal if and only if L is very ample.*
  (b) *If $n_2 \geqslant 7$ and A is generic ($\mathrm{NS}(A) \cong \mathbb{Z}$) then φ_L is a projectively normal embedding.*
  (c) *If $n_2 > 8$ and A is not isogenous to a product of elliptic curves then φ_L is a projectively normal embedding.*
(4) *If L has type $(1, n)$ then the induced map $\varphi_L : A \to \mathbb{P}^{n-1}$ is a projectively normal embedding if and only if L is very ample and $n \geqslant 7$.*

6.3. **Algebraic Stacks and Embeddings not over Algebraically Closed Fields.** On the other hand, Cox rings makes sense with more general base rings that $\mathbb{C}$ which was done for this document. Algebraic stacks and embeddings over rings which are not algebraically closed fields, say free modules for example, are two immediate examples of ways these Petri equations or syzygy theory might also be generalized.

In this section some results of [26] about Cox rings of stacky curves are stated by way of demonstrating that the kind of tools employed in this paper already have some generalizations to curves where to paraphrase Green, the intrinsic geometry of the curve is actually considered. Luckily this also comes with some tricks like Noether's theorem being opened up beyond the complex embeddings Green and Lazarsfeld describe.

The generalization of Noether's theorem in [26] is done over a general field k but since this kind of Cox ring computation works for less friendly base rings, the particular things about being a field which make the following work are of great importance. Though the proof of the theorem which immediately follows is done over an algebraically closed field, Voight and DZB emphasize that the surjectivity of the maps which make the embedded curve projectively normal per the classical statement of Noether, is a statement in linear algebra.

Theorem 6.30. [26, 3.2.1] *Let X be a genus $g \geqslant 2$ curve over a field $\mathbb{F}$ and let E, E' be effective divisors on X. Then the mulitplication*

$$H^0(X, K + E) \otimes H^0(X, K + E') \to H^0(X, 2K + E + E')$$

is surjective if and only if one of the following holds

(1) *X is not hyperelliptic, $g \geqslant 3$ and* $\deg E = \deg E' = 0$
(2) *$E \not\sim E'$ (over $\overline{\mathbb{F}}$) or not both of E and E' are hyperelliptic fixed and* $\deg E = \deg E' = 2$
(3) $\deg E \geqslant 3$ *and either* $\deg E' = 0$ *or* $\deg E' \geqslant 2$.

Of course the punchline of [26] is important here as well, not only because it demonstrates a respect for the intrinsic geometry of modular curves, or log stacky orbifold curves, but also as a kind of motivation for all of this work.
The Cox ring of a modular curve has a unique structure since the graded pieces $R_d = H^0(X_\Gamma, \Omega^d)$ of the canonical ring are the cusp forms $S_{2d}(\Gamma)$ and $H^0(X, \Omega(\Delta)^{\otimes d}) = M_{2d}(\Gamma)$. With this idea, the canonical ring of the log curve (X, Δ) for Δ the divisor of cusps for Γ leads to some interesting classical examples.

Example. Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and let $X(1) = \Gamma\backslash\mathcal{H}^*$ with $\Delta = \infty$. Then $R_{K+\Delta}(X(1)) = \mathbb{C}[E_4, E_6]$ so even though the log curve $(X(1), \Delta)$ has genus 0 and hence a trivial canonical ring as a Riemann surface, the underlying scheme over $\mathbb{C}$, $X(1) = \mathrm{Proj} R_{K+\Delta}(X(1))$ so the log curve must be treated as a curve with an ample canonical divisor.

Theorem 6.31. [26] *Let (X, Δ) be a tame log stacky curve over a field k with signature $(g; e_1, \cdots, e_r, \delta)$. Let $e = \max\{1, e_1, \cdots, e_r\}$. Then $R(X, \Delta) = \bigoplus_{d\geqslant 0} H^0(X, \Omega(\Delta)^{\otimes d})$ is generated as a k-algebra in degree at most $3e$ with relations among those generators of degree at most $6e$.*

Better yet would be to make these computations in the Drinfeld setting, so to consider the upper half plane in the function field setting, where the algebraically closed base field over which the "scheme stuff" happens, is infinite dimensional. The air quotes are not intended to be cheeky here either, but rather to indicate that actually something totally nontrivial happens when considering the moduli space of Drinfeld modules as an orbifold curve. The natural instinct of the arithmetic geometer is to use GAGA principles, and at worst stacks, to deal with things like elliptic curves with level structure by forming their moduli space, and although Drinfeld modules have a corresponding lattice quotient theory, and even a level structure, there is an obstruction to what maybe has to be called classical stack theory now, in the Drinfeld setting. Some new version of a stacky curve needs to be defined to be able to use tech such as [26] to compute the generators and relations for the canonical ring of a congruence subgroup in the Drinfeld setting.

To this end and by way of connecting this section to the last one about surfaces there is one more kind of result which is worth including here: the theorems of Aaron Landesman, Peter Ruhm, and Robin Zhang.

Theorem 6.32. [15, 1.1.1] *Let $D = \sum_{i=0}^{n} \alpha_i D_i \in \mathrm{Div}\,\mathbb{P}^r \otimes_{\mathbb{Z}} \mathbb{Q}$ for some $\alpha_i = \frac{c_i}{k_i} \in \mathbb{Q}_{>0}$ in reduced form and each $D_i \in \mathrm{Div}\,\mathbb{P}^r$ an integral divisor. The section ring $R(\mathbb{P}^r, D)$ is generated in degree at most $\max_{0\leqslant i\leqslant n} k_i$ with relations generated in degree at most $2\max_{0\leqslant i\leqslant n} k_i$.*

Theorem 6.33. [15, 1.1.2] *Let $D = \sum_{i=0}^{n} \alpha_i D_i \in \mathrm{Div}\,\mathbb{P}^r \otimes_{\mathbb{Z}} \mathbb{Q}$ for some $\alpha_i = \frac{c_i}{k_i} \in \mathbb{Q}$ in reduced form. Let $l_i = \mathrm{lcm}_{0\leqslant j\leqslant n,\ j\neq i}\, k_j$ and let $a_i = \deg D_i$. Let $\mathbb{P}^r \cong \mathrm{Proj} k[x_0, \cdots, x_r]$ and $f_i \in k[x_0, \cdots, x_r]$ be such that $D_i = V(f_i)$. If $\{f_0, \cdots, f_n\}$ contains a basis for $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$ then $R(\mathbb{P}^r, D)$ is generated in degrees at most $\omega = \sum_{i=0}^{n} l_i a_i$ with relations generated in degrees at most*

$$\max\left(2\omega, \frac{\max_{0\leqslant i\leqslant n} a_i}{\deg D} + \omega\right).$$

References

[1] Marco Andreatta and Edoardo Ballico. Projectively normal adjunction surfaces. *Proc. Amer. Math. Soc.*, 112(4):919–924, 1991.

[2] Enrico Arbarello, Maurizio Cornalba, and Phillip A. Griffiths. *Geometry of algebraic curves. Volume II*, volume 268 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011. With a contribution by Joseph Daniel Harris.

[3] David A. Cox, John Little, and Donal O'Shea. *Using algebraic geometry*, volume 185 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2005.

[4] David S. Dummit and Richard M. Foote. *Abstract algebra*. John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2004.

[5] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[6] The Encyclopedia of Math Authors. Canonical imbedding. `https://encyclopediaofmath.org/wiki/Canonical_imbedding`, 2020.

[7] Luis Fuentes García. Projective normality of abelian surfaces of type $(1, 2d)$. *Manuscripta Math.*, 114(3):385–390, 2004.

[8] James McKernan. Finite generation of the canonical ring. Finite generation of the canonical ring. `https://math.mit.edu/~mckernan/Talks/finite.pdf`.

[9] Antonella Grassi. Birational geometry old and new. *Bull. Amer. Math. Soc. (N.S.)*, 46(1):99–123, 2009.

[10] Mark Green and Robert Lazarsfeld. A simple proof of Petri's theorem on canonical curves. In *Geometry today (Rome, 1984)*, volume 60 of *Progr. Math.*, pages 129–142. Birkhäuser Boston, Boston, MA, 1985.

[11] Mark L. Green. Koszul cohomology and the geometry of projective varieties. II. *J. Differential Geom.*, 20(1):279–289, 1984.

[12] Yi Hu and Sean Keel. Mori dream spaces and GIT. volume 48, pages 331–348. 2000. Dedicated to William Fulton on the occasion of his 60th birthday.

[13] Janos Kollar. Families of varieties of general type. `https://web.math.princeton.edu/~kollar/book/modbook20170720.pdf`, 2021.

[14] Janos Kollar and Sander Kovacs. Singularities of the minimal model program. `https://www.cambridge.org/core/books/singularities-of-the-minimal-model-program/BFD87F262E5174A425F12AD85270E0AA`, 2021.

[15] Aaron Landesman, Peter Ruhm, and Robin Zhang. Section rings of $\mathbb{Q}$-divisors on minimal rational surfaces. *Math. Res. Lett.*, 25(4):1329–1357, 2018.

[16] David Mumford. *The red book of varieties and schemes*, volume 1358 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, expanded edition, 1999. Includes the Michigan lectures (1974) on curves and their Jacobians, With contributions by Enrico Arbarello.

[17] Roberto Pignatelli. Surfaces of general type. `http://www.crm.sns.it/media/event/345/gt.pdf`, 2015. [Accessed 11 July 2022].

[18] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu/tag/00DM`, 2018.

[19] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu/tag/00CV`, 2018.

[20] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu/tag/0120`, 2018.

[21] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu/tag/0BF9`, 2018.

[22] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu/tag/01QN`, 2018.

[23] Kenji Ueno. Introduction to classification theory of algebraic varieties and compact complex spaces. In *Classification of algebraic varieties and compact complex manifolds*, pages 288–332. Lecture Notes in Math., Vol. 412. 1974.

[24] Ravi Vakil. lecture notes. `https://math.stanford.edu/~vakil/0506-216/216class3940.ps`, 2002.

[25] Ravi Vakil. lecture notes. `https://math.stanford.edu/~vakil/0506-216/216class5152.pdf`, 2006.

[26] John Voight and David Zureick-Brown. The canonical ring of a stacky curve. *Mem. Amer. Math. Soc.*, 277(1362):v+144, 2022.