Example: Interpreting Predicate Formulas

Given a structure $\mathcal{A}$ equipped with the following universe and interpretation function, as well as equality over the naturals and the universe:

$$
\begin{aligned}
\mathcal{U}_{\mathcal{A}} &\triangleq \{f_1, \ldots, f_{10}\} \\
[\![g^1]\!]_{\mathcal{A}} &\triangleq \text{returns the count of legs for the input} \\
[\![P^1]\!]^{\mathcal{A}} &\triangleq \text{is a table} \\
[\![Q^2]\!]^{\mathcal{A}} &\triangleq \text{lamp is on the table} \\
[\![R^2]\!]_{\mathcal{A}} &\triangleq \text{the artisan crafted the furniture} \\
[\![a]\!]_{\mathcal{A}} &\triangleq \text{Ron Swanson}
\end{aligned}
$$

The tables with the lamps on them are crafted by Ron Swanson.

$$\forall x\,((P(x) \land Q(x)) \rightarrow R(a, x))$$

All tables have four legs.

$$\forall x\,(P(x) \rightarrow g(x) = 4)$$

Substitution

This can be tricky (both by hand and in code).

$$\mathcal{A}_{[x/x_0]}(F) \triangleq \text{“replace every free instance of } x \text{ with } x_0 \text{ in } F.\text{”}$$

Note: there are many ways to write this (including $\mathcal{A}_{[x_0/x]}(F)$!). (Should be clear from context; do not be afraid to ask.)

Easiest method: starting from the innermost bound variables, rename:

$$P(x) \vee \forall x \left(\exists x \left(P(x) \to Q(x) \right) \wedge \forall x (R(x)) \wedge S(x) \right)$$

Equivalence and satisfiability of predicate logic

Recall:

$$F \equiv G \triangleq [\![F]\!]_{\mathcal{A}_1} = [\![G]\!]_{\mathcal{A}_1} \land \cdots \land [\![F]\!]_{\mathcal{A}_n} = [\![G]\!]_{\mathcal{A}_n}$$

Two formulas F and G are equivalent if and only if they evaluate to the same truth value for all suitable assignments.

Two formulas F and G are equivalent if and only if they evaluate to the same truth value for all *suitable structures*?

$$\begin{aligned} F \quad & \triangleq \forall x\,(P(x, f(x))) \\ & \quad \land \forall y \neg P(y, y) \\ & \quad \land \forall u \forall v \forall w\,(P(u, v) \land P(v, w) \to P(u, v)) \\ G \quad & \triangleq \text{ some formula containing the same set of symbols} \end{aligned}$$

Easy mode: the formulas have the same surface string (syntax!).
Hard mode: the formula is occluded.

Finding SAT vs. Evaluating Predicate Logic

Propositional logic: find a mapping from *propositions* to truth values to makes the formula true.
Predicate logic: find a structure where the mapping from *interpreted variables* to truth values makes the formula true.

Logic: searching for a universe, emphasis on core mathematical truths.
AI: often have a universe in mind, really: SAT over formula + structure

Even with this information, exhaustive search is hard.

For certain tasks, there is an easier way!

From rewrite rules to inference

Recall:

$\neg\neg F = F$ (double negation)

$\neg(F \wedge G) = \neg F \vee \neg G$ (deMorgan's)

$\neg(F \vee G) = \neg F \wedge \neg G$ (deMorgan's)

$F \wedge (G \vee H) = (F \wedge G) \vee (F \wedge H)$ (distributive)

$F \vee (G \wedge H) = (F \vee G) \wedge (F \vee H)$ (distributive)

$F \rightarrow G = \neg F \vee G$

New ones for predicate logic:

$\neg\forall x(F(x)) = \exists x(\neg F(x))$ (deMorgan's)

$\neg\exists x(F(x)) = \forall x(\neg F(x))$ (deMorgan's)

How do we create new knowledge from what we already know, using only syntax, not semantics?

Sequents vs. Implications (Inference vs. Derivations)

$$F_1, \ldots, F_n \vdash G$$

$$(F_1 \land \ldots \land F_n) \to G$$

Example Scenario: I have a proof of $F \to G$ and I a proof of G, but I want a proof of G, *and I can't break apart F or G to get it*. How do I produce such a proof?

We can think about a "proof" as datum that ensures the path of our reasoning is correct.

Inference rules

Inference rules are if-statements that let us combine old information into new information.

- and elimination
- and introduction
- or introduction
- implication elimination
- bottom elimination
- not elimination

Inference rules

Some inference rules require the notion of scope.

implication introduction

or elimination

not introduction

Inference rules with quantifiers always require scope

Natural deduction: inference rules + scope

For a sequent $F_1, \ldots, F_n \vdash G$, $F_1, \ldots, F_n$ are always in scope, e.g.,

$$\forall(P(x) \to Q(x)), \exists x P(x) \vdash \exists Q(x)$$

Clicker Question

Which of the following lines is incorrect?

Next week, next unit

Next week: Remainder of unit 1 (logic): programming and applications-focused.

Preview of next unit: We will put a pin this for now, *but*...much of AI classically focuses on search and check. This is the "Hard" part of AI; the "hard" part is often the encoding (knowledge representation).