e.g. Find the value of k st the function is continuous:

$$g(x) = \begin{cases} \dfrac{2x^2-x-15}{x-3}, & x \neq 3 \\ kx-1, & x=3 \end{cases}$$

We need to choose k so that these two separate pieces

$$\begin{cases} * & \dfrac{2x^2-x-15}{x-3} \\ ** & kx-1 \end{cases}$$

agree (match up) where they are "pasted" together

First, lets find the limit as $x \to 3$ of (*)

$$\lim_{x \to 3} \frac{2x^2-x-15}{x-3} = \frac{(2x+5)(x-3)}{(x-3)} = 2x+5 \quad (x \neq 3)$$

$$= 2 \cdot 3 + 5 = 11$$

Now, when $x=3$

$$kx-1 = 3k-1$$

In order for these to agree, we must set them equal and solve for k:

$$2x+5 \Big|_{x=3} = kx-1 \Big|_{x=3}$$

$$\Rightarrow 6+5 = 3k-1$$

$$\Rightarrow 11 = 3k-1$$

$$\Rightarrow k = \frac{12}{3} = \boxed{4}$$

$(C)\ \psi(x) = e^{\sqrt{2x-3}}$

$2x-3 \geq 0$

$x \geq 3/2$

e.g. Find all points of discontinuity for the function

(A) $f(x) = \frac{4x-3}{2x-7}$: Rational $\rightarrow$ $2x-7=0 \Rightarrow x = 7/2$

(B) $g(x) = e^{2x-3}$: exponential $\rightarrow$ No discontinuities

e.g. Find all values of x where the function is continuous

$$f(x) = \begin{cases} x+1 & \text{if } x<1 \\ x^2-3x+4 & \text{if } x \in [1,3] \\ 5-x & \text{if } x>3 \end{cases}$$

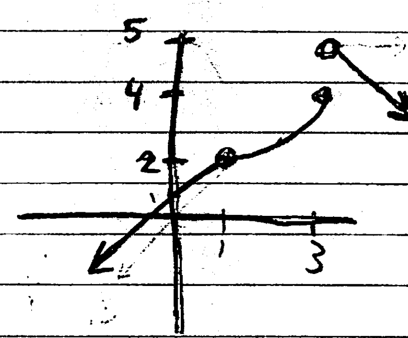
- Observe f is continuous at $(-\infty,1)$, $(1,3)$, $(3,\infty)$ at the very least, since it is equal to various polynomial functions at these values.
- However, we must check the places where f is "pasted together" (the endpoints)

$\underline{x=1}$ $f(1) = 1^2-3+4 = 2$

$\lim_{x\to 1^-} f(x) = \lim_{x \to 1^-} x+2 = 2$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2-3x+4 = 1^2-3+4 = 2$

So f is indeed continuous at $x=1$



$\underline{x=3}$ $f(3) = 3^2-3\cdot 3+4 = 9-9+4 = 4$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 5-x = 5-3 = 2$ $\neq$

$\lim_{x\to 3^-} f(x) = \lim_{x \to 3^-} 3^2-3\cdot3+4 = 9-9+4 = 4$

So f is NOT continuous at $x=3$

$\Rightarrow$ f is continuous on $\boxed{(-\infty,3)\cup(3,\infty)}$

§3.3 : Rates of Change

- One of the main applications of Calc is determining how one variable changes in relation to the other
- For lines, this is easy, the rate of change is the slope.
- We can approximate this for any function $f(x)$

Average Rate of Change

- The average rate of change of a function $f(x)$ with resp to x as x changes from a to b is:

$$f_{avg} = \frac{f(b) - f(a)}{b - a}$$

e.g. Suppose 89.7% of households had landline phones in 2005, while in 2009 this number reduced to 73.5%. Determine the average rate of change in the percent of landlines in American households per year over 2005 - 2009:

- Plug + Chug:

$$f_{avg} = \frac{73.5 - 89.7}{2009 - 2005} = \frac{-16.2}{4} = -4.05$$

Decline (negatives) is roughly 4.05% per year.

Instantaneous Rate of Change

- what if we want to know the exact speed of a moving vehicle at a certain moment in time when we have a function, $f(x)$ for vehicles position
- How do we do this? We know the average rate of change
- Using our formula for average rate of change lets take a, b to be very close together $\frac{f(b)-f(a)}{b-a}$

How close can we make b and a? (The closer the more accurate)

Consider a function $f(x)$ that we want to find the instantaneous velocity at point $x=a$. Let h be a very small number "close" to a. Then:

$$\frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$$

is closer to our desired instantaneous value as h gets small.

For smaller and smaller values of h we get a more accurate approximation. How do we make it arbitrarily small? LIMITS!

Instantaneous Rate of Change

The instantaneous rate of change for a function f when $x=a$ is:

$$\boxed{\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}}$$

provided the limit exists

Alternate:

$$\lim_{b \to a} \frac{f(b)-f(a)}{b-a}$$

e.g. Find the instantaneous rate of change of $f(x) = 3x^2$ at $x=10$

here: $f(10) = 3 \cdot 10^2 = 300$; $f(10+h) = 3(10+h)^2$
$= 3(100+20h+h^2)$
$= 300+60h+3h^2$

Plug it in:

$$\lim_{h\to 0} \frac{f(10+h)-f(10)}{h} = \lim_{h\to 0} \frac{\cancel{300}+60h+3h^2-\cancel{300}}{h}$$

$$= \lim_{h\to 0} \frac{60h+3h^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(60+3h)}{\cancel{h}} = \lim_{h\to 0} 60+3h = \boxed{60}$$

e.g. Distance in feet of an object from a starting Point is given by $s(t) = 2t^2-5t+40$, where t is time in seconds

(A) Find avg velocity between $t=2$ and $t=4$ seconds

$$f_{avg} = \frac{f(t=4)-f(t=2)}{4-2} = \frac{[2(4)^2-5(4)+40]-[2(2)^2-5(2)+40]}{2}$$

$$= \frac{[32-20+40]-[8-10+40]}{2} = \frac{52-38}{2} = \frac{14}{2} = \boxed{7 \text{ ft/sec}}$$

(B) Instantaneous velocity at $t=4$ seconds:

$$\lim_{h\to 0} \frac{f(4+h)-f(4)}{h} = \lim_{h\to 0} \frac{[2(4+h)^2-5(4+h)+40]-[2(4^2)-5(4)+40]}{h}$$

$$= \lim_{h\to 0} \frac{2(16+8h+h^2)-20-5h+40-52}{h}$$

$$= \lim_{h\to 0} \frac{2h^2+(16-5)h+(32-20+40-52)}{h} = \lim_{h\to 0} \frac{2h^2+11h}{h} = \lim_{h\to 0} \frac{\cancel{h}(h+11)}{\cancel{h}} = \boxed{11 \text{ ft/sec}}$$

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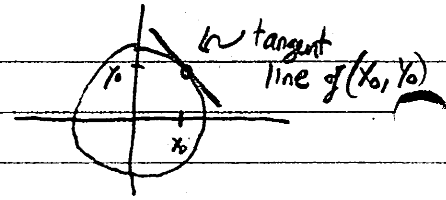
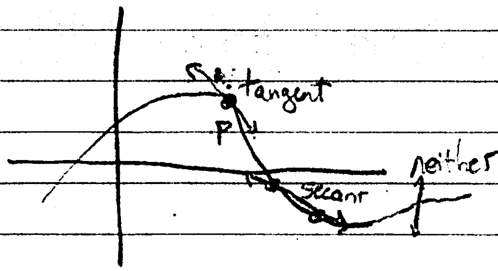
§3.4: Definition of the Derivative



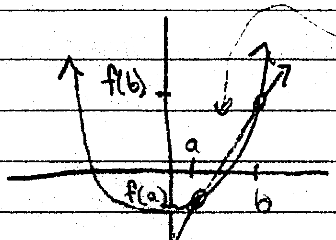
- In geometry a tangent line to a circle is a line that touches the circle at exactly 1 point.

- Similarly, for any function, a tangent line at a point P is a line that will touch the function at P but not at any points "nearby"



with the function "bouncing off the line at P"

- When we find the average rate of change of a function, what we actually find is the slope of a "secant" line through the function. (the secant here passes through $(a, f(a))$, $(b, f(b))$)



secant line with slope

$$m = \frac{f(b) - f(a)}{b - a}$$

- We already know how to find the slope of a tangent line at a point for a function!

Slope of a Tangent Line

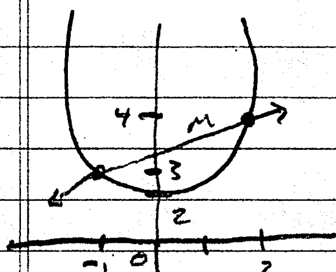
The tangent line of the graph $y = f(x)$ at the point $(a, f(a))$ is the line through this point with slope.

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

; provided the limit exists. If the limit does NOT exist, then there is no tangent line at this point.

E.g.| Consider the function $f(x) = x^2+2$

(A) Find the slope & equation of the secant line passing through the graph of f at $x=-1$, and $x=2$



Find the slope and use point slope form:

$$m = \frac{\Delta y}{\Delta x} = \frac{f(2)-f(-1)}{2-(-1)} = \frac{6-3}{3} = \boxed{1}$$

Lets use the point $(-1,3)$ and $m=1$

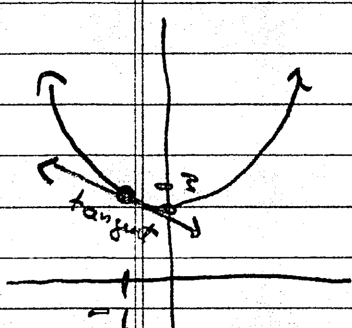
$$y - y_1 = m[x - x_1] \quad \text{(Point Slope)}$$

$$\Rightarrow y - 3 = 1[x-(-1)] \Rightarrow \boxed{y = x+4}$$

(B) Find the slope and equation of the tangent line at $x=-1$

Use the equation for instantaneous rate of change w/ $a=-1$

$$\Rightarrow \text{Slope of tangent line} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$



$$= \lim_{h\to 0} \frac{[(-1+h)^2+2] - [(-1)^2+2]}{h}$$

$$= \lim_{h\to 0} \frac{1-2h+h^2+2-3}{h} = \lim_{h\to 0} \frac{-2h+h^2}{h}$$

$$= \lim_{h\to 0} \frac{h(-2+h)}{h} = \lim_{h\to 0} -2+h = -2+0 = \boxed{-2}$$

We find the equation of the line using point slope form:
$(m=-2,\ x_0=-1,\ y_0=3)$

$$\Rightarrow y-3 = -2(x+1) \Rightarrow \boxed{y=-2x+1}$$

The Derivative:

The derivative of the function $f(x)$ at x is defined as:

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$$ provided this limit exists.

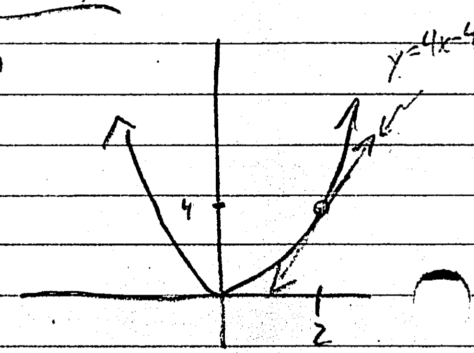
e.g.) Find the derivative

(A) $f(x) = x^2$

$f(x+h) = (x+h)^2 = x^2 + 2hx + h^2$

$\Rightarrow f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{x^2+2hx+h^2-x^2}{h}$

$= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = 2x+0 = \boxed{2x}$



(B) Calculate and interpret (graphically) $f'(2)$

$f'(2) = 2\cdot 2 = 4$

$\Rightarrow$ eqn of tangent line at $x=2$ is: (at pt $(2,4)$, $m=f'(2)$)

$y-4 = f'(2)(x-2) = 4(x-2)$

$\Rightarrow \boxed{y = 4x-4}$

Finding $f'(x)$ from the definition of the derivative

1) Find $f(x)$

2) find and simplify $f(x+h)-f(x)$

3) divide by h to get $\frac{f(x+h)-f(x)}{h}$

4) Let $h\to 0$ i.e.

find $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$ (if the limit exists) e.g.] Let $f(x) = 2x^3 + 4x$. Find $f'(x)$, $f'(2)$, $f'(-3)$

Following those steps: ① $f(x+h) = 2(x+h)^3 + 4(x+h)$

$$= 2(x^3 + 3x^2h + 3xh^2 + h^3) + 4(x+h)$$
$$= 2x^3 + 6x^2h + 6xh^2 + 2h^3 + 4x + 4h$$

② $f(x+h) - f(x) = 6x^2h + 6xh^2 + 2h^3 + 4h$

③,④ $f'(x) = \lim_{h\to 0} \frac{\cancel{h}(6x^2 + 6xh + 2h^2 + 4)}{\cancel{h}} = 6x^2 + 0 + 0 + 4$

$$= \boxed{6x^2 + 4}$$

Plug & Chug

$f'(2) = 6(2)^2 + 4 = 6 \cdot 4 + 4 = \boxed{28}$

$f'(-3) = 6(-3)^2 + 4 = 6 \cdot 9 + 4 = \boxed{58}$

e.g.] Let $f(x) = \frac{4}{x}$; find $f'(x)$

$f(x+h) = \frac{4}{x+h}$; $\Rightarrow f(x+h) - f(x) = \frac{4}{x+h} - \frac{4}{x} = \frac{4x - 4(x+h)}{(x+h)(x)}$

$\Rightarrow f'(x) = \lim_{h\to 0} \frac{\frac{4x - 4x - 4h}{(x+h)(x)}}{h} = \lim_{h\to 0} \frac{\cancel{h}(-4)}{(x+h)(x)} \cdot \frac{1}{\cancel{h}} = \frac{-4}{(x+0)(x)} = \boxed{\frac{-4}{x^2}}$

pg 172 e.g.] $w(x) = \sqrt{x} + 40$

$w(x+h) = \sqrt{x+h} + 40 \Rightarrow w(x+h) - w(x) = \sqrt{x+h} + \cancel{40} - \sqrt{x} - \cancel{40}$

To divide by h we must rationalize the numerator:

$\Rightarrow f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{\sqrt{x+h} + \sqrt{x}} = \lim_{h\to 0} \frac{\cancel{x} + h - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})}$

$$= \frac{1}{\sqrt{x+0} + \sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}} = \frac{1}{2}x^{-1/2}$$

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(CH3 pg 174)

Existence of the Derivative

The derivative of a function exists when f satisfies all of the following conditions at a point.

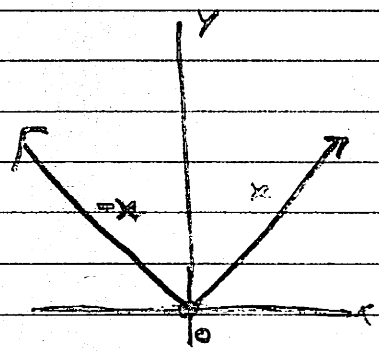
1) f is continuous
2) f is smooth
3) f does NOT have a vertical tangent line.

The derivative of f does NOT exist when any of the following conditions are true for a function at a point.

1. f is discontinuous
2. f has a sharp corner
3. f has a vertical tangent line

e.g.) Find where the function does NOT have a derivative.

(a) $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$ We see a sharp corner at $x=0$, so NOT continuous there.



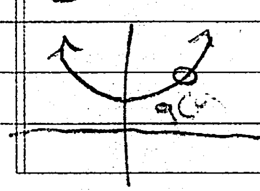
If we tried to take the derivative seperately of each component we get:

$$\lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0} \frac{h}{h} = 1, \text{ while}$$

$$\lim_{h\to 0} \frac{-(x+h)+x}{h} = \lim_{h\to 0} \frac{-h}{h} = -1$$

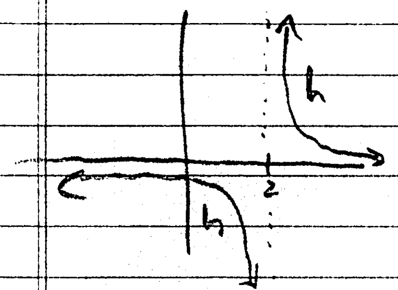
Hence, not smooth, so NOT differentiable at $x=0$.

(b) $g(x) = \frac{x^2-4}{x-2}$; we see $g(x) = x+2$ with a hole at $x=2$ $(2,4)$



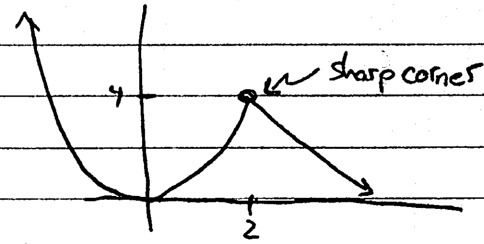
So it is NOT continuous here and its derivative does not exist (here!)

(C) $h(x) = \frac{1}{x-2}$



Here we have an asymptote at $x=2$, so the function is NOT defined at $x=2$, Hence NOT differentiable at $x=2$

(D) $S(x) = \begin{cases} x^2 & ; x \le 2 \leftarrow \stackrel{\Delta}{=} S_1(x) \\ -x+6 & ; x>2 \leftarrow \stackrel{\Delta}{=} S_2(x) \end{cases}$



Is the function continuous at $x=2$? Check pieces:

$S_1(2) = 2^2 = 4$; $S_2(2) = -2+6 = 4$, so yes continuous

But we see a sharp corner. Let's check derivatives of the components:

$$S_1'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{\cancel{x^2} + 2hx + h^2 - \cancel{x^2}}{h} = \lim_{h\to 0} 2x+h = \boxed{2x}$$

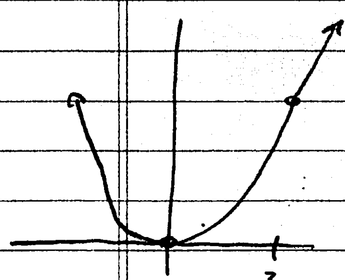
$\neq$

$$S_2'(x) = \lim_{h\to 0} \frac{-(x+h)+6-(-x+6)}{h} = \lim_{h\to 0} \frac{-h}{h} = \boxed{-1}$$

Since these derivatives don't match up, the derivative does NOT Exist at $x=2$, specifically since $S_1'(2) = 4 \neq -1 = S_2'(2)$

(E) $t(x) = \begin{cases} x^2 & ; x \le 2 \leftarrow t_1(x) \\ 4x-4 & ; x > 2 \leftarrow t_2(x) \end{cases}$ Continuous at $x=2$?

$t_1(2) = 4 \stackrel{?}{=} t_2(2) = 4 \cdot 2 - 4 = 4$ ✓



Looks smooth at $x=2$, lets make sure:

$t_1'(x) = \boxed{2x}$

$$t_2'(x) = \lim_{h\to 0} \frac{4(x+h)-4-(4x-4)}{h} = \lim_{h\to 0} \frac{4h}{h} = \boxed{4}$$

And $t_1'(2) = 2 \cdot 2 = 4 \stackrel{?}{=} t_2'(2) = 4$ ✓ So it is indeed smooth here.

It's important that the slope of the tangent lines match up here.

§4.1 : Techniques for finding Derivatives

Notation:

The derivative of $y=f(x)$ may be written as:

$f'(x)$, $\frac{dy}{dx}$, $\frac{df}{dx}$, $\frac{d}{dx}[f(x)]$, $D_x[f(x)]$, $\left\{ \overset{\text{other}}{\dot{f}(x),\ f_x(x)} \right\}$

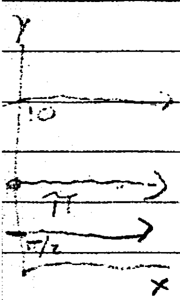
i.e. We found the derivative of $f(x)=2x^3+4x$ to be $f'(x)=6x^2+4$

so $f'(x)=\frac{df}{dx}=D_x(2x^3+4x)=\frac{d}{dx}(2x^3+4x)=6x^2+4$

This means the derivitive of f with respect to x

What's the derivative of a constant function?

$f(x)=C$ for $c\in\mathbb{R}$. Using limit definition:



$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{C-C}{h}=\lim_{h\to 0}\frac{0}{h}=0$

Constant Rule: If $f(x)=c$ for $c\in\mathbb{R}$ then $f'(x)=0$

Show Derivative of $f(x)=x^n$ Table (pg 199)

Power Rule: If $f(x)=x^n$, for any $n\in\mathbb{R}$, then

$\boxed{f'(x)=nx^{n-1}}$ also if $a\in\mathbb{R}$ then $g(x)=ax^n$ $\Rightarrow g'(x)=(n\cdot a)x^{n-1}$

e.g. Find the derivative w/ the Power rule:

(A) $f(x)=4x^5 \Rightarrow f'(x)=(5\cdot 4)x^{5-1}=20x^4$

(B) $y=3/x^2=3x^{-2}\Rightarrow \frac{dy}{dx}=(-2\cdot 3)x^{-2-1}=-6x^{-3}=\frac{-6}{x^3}$

(C) $D_x[3x^{2/3}]=(\frac{2}{3}\cdot 3)x^{\frac{2}{3}-1}=2x^{-1/3}=\frac{2}{\sqrt[3]{x}}$

(D) $\frac{d}{dz}(\pi\sqrt{z})=\frac{d}{dz}(\pi z^{1/2})=(\frac{1}{2}\cdot\pi)z^{\frac{1}{2}-1}=\frac{\pi}{2}z^{-1/2}=\frac{\pi}{2\sqrt{z}}$