Math 395 - Fall 2021
Qual problem set 5

This homework is "due" on Monday October 4 at 11:59pm.

You may also (in addition to or instead of turning this in as a homework, your choice) use this assignment as a quiz. In this case, give yourself one hour to solve two of these problems completely.

1. Let G be a group of order $10,989 = 3^3 \cdot 11 \cdot 37$.
   (a) Compute the number n_p of Sylow p-subgroups permitted by Sylow's Theorem for each of $p = 3, 11$ and 37; for each of these n_p give the order of the normalizer of a Sylow p-subgroup.
   (b) Show that G contains either a normal Sylow 37-subgroup or a normal Sylow 3-subgroup.
   (c) Explain briefly why (in all cases) G has a normal Sylow 11-subgroup.
   (d) Deduce that the center of G is nontrivial.

2. Assume that G is a *simple* group of order $4851 = 3^2 \cdot 7^2 \cdot 11$.
   (a) Compute the number n_p of Sylow p-subgroups permitted by Sylow's Theorem for each of $p = 3, 7$, and 11; for each of these n_p give the order of the normalizer of a Sylow p-subgroup.
   (b) Show that there are distinct Sylow 7-subgroups P and Q such that $\#P \cap Q = 7$.
   (c) For P and Q as in (b), let $H = P \cap Q$. Explain briefly why 11 does not divide $\#N_G(H)$.
   (d) Show that there is no simple group of this order. (Hint: How many Sylow 7-subgroups does $N_G(H)$ contain, and is this permissible by Sylow?)

3. Let G be a finite group with the property that the centralizer of every nonidentity element is an *abelian* subgroup of G. (Such a group is called a *CA*-group.)
   (a) Prove that every Sylow p-subgroup of G is abelian, for every prime p.
   (b) Prove that if P and Q are distinct Sylow subgroups of G, then $P \cap Q = 1$.

4. Let G be a finite group, let p be a prime and let $P \in Syl_p(G)$. Assume that P is abelian.
   (a) Prove that two elements of P are conjugate in G if and only if they are conjugate in $N_G(P)$.
   (b) Prove that $P \cap gPg^{-1} = 1$ for every $g \in G - N_G(P)$ if and only if $P \trianglelefteq C_G(x)$ for every nonidentity element $x \in P$.