

Math 395 - Fall 2021
Final Exam

The Final Exam will be graded as follows:

10/10 for six complete problems

9.5/10 for four complete problems and substantial progress on the other two problems

8.5/10 for nine complete lettered parts

6/10 for six complete lettered parts

3/10 for three complete lettered parts

You do not need to type up your answers.

Section A: Group Theory

1. Let G be a finite group and let p be a prime. Assume G has a normal subgroup H of order p .
 - (a) Prove that H is contained in every Sylow p -subgroup of G .
 - (b) Prove that if p is the smallest prime dividing the order of G , then H is contained in the center of G .
 - (c) Prove that if G/H is a nonabelian simple group, then H is contained in the center of G .
2.
 - (a) Please state the Class Equation for finite groups. (Hint: It is located in the section on Groups Acting on Themselves by Conjugation.)
 - (b) Prove that if G is a p -group, then G has nontrivial center.
 - (c) Prove that if G is a p -group, then G is solvable.
3. Let G be a group of order 6545 (note that $6545 = 5 \cdot 7 \cdot 11 \cdot 17$).
 - (a) Compute the number n_p of Sylow p -subgroups permitted by Sylow's Theorem for $p = 5$ and $p = 17$ (only).
 - (b) Let P_5 be a Sylow 5-subgroup of G . Prove that if P_5 is not normal in G , then $N_G(P_5)$ has a normal Sylow 17-subgroup. (Hint: Use that $P_5 \trianglelefteq N_G(P_5)$.)
 - (c) Deduce from (b) and (a) that G has a normal Sylow p -subgroup for either $p = 5$ or $p = 17$.

Section C: Field Theory

4. Let E be the splitting field in \mathbb{C} of the polynomial $p(x) = x^6 + 3x^3 + 3$ over \mathbb{Q} , and let α be any root of $p(x)$ in E .
- (a) Find $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.
 - (b) Show that $\alpha^3 + 1 = \omega$ is a primitive cube root of unity. Describe the roots of $p(x)$ in terms of radicals involving rational numbers and ω .
 - (c) Assume that $E \neq \mathbb{Q}(\alpha)$, and prove $[E : \mathbb{Q}] = 18$.
(Hint: Show first that $\mathbb{Q}(\beta)$ is a Galois extension of $\mathbb{Q}(\omega)$ of degree 3, for every root β of $p(x)$.)
 - (d) Again assume $E \neq \mathbb{Q}(\alpha)$, and prove that E contains a *unique* subfield F with $[F : \mathbb{Q}] = 2$.
5. (a) Give an example of an extension K/\mathbb{Q} that is Galois with Galois group C_4 and prove that this is such an example.
- (b) Give an example of an extension K/\mathbb{Q} that is Galois with Galois group $C_2 \times C_2$ and prove that this is such an example.
- (c) Give an example of an extension K/\mathbb{Q} that is Galois with Galois group D_4 and prove that this is such an example.
6. Let K be the splitting field of $x^{61} - 1$ over the finite field \mathbb{F}_{11} .
- (a) Find the degree of K over \mathbb{F}_{11} .
 - (b) Draw the lattice of all subfields of K . (You need not give generators for these subfields.)
 - (c) How many elements $\alpha \in K$ generate the multiplicative group K^\times ?
 - (d) How many primitive elements are there for the extension K/\mathbb{F}_{11} ? (In other words, how many β are there such that $K = \mathbb{F}_{11}(\beta)$?)