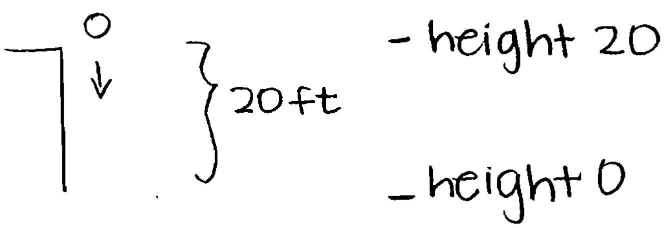
MATH 320

Supplementary Problems (Sections 1.2, 1.3)

Section 1.2

#33. The key physics fact necessary to solve this problem is that the downwards acceleration due to gravity on a given planet is constant.

To solve the problem, we must first find this constant:



Let $y(t)$ denote the position (height) of the ball at time t. Then:

$y(0) = 20$ ← starts 20 ft high

$y(2) = 0$ ← hits the ground after 2 seconds

$y'(0) = 0$ ← the problem doesn't say that the ball had any initial velocity, so this means it started from rest.

The acceleration is constant: $y''(t) = k$.

$y''(t) = k$ or $\frac{d^2y}{dt^2} = k$ is the differential equation

we must solve:

$$y'(t) = \int (y''(t))\,dt = \int k\,dt = kt + C$$

$$y(t) = \int (y'(t))\,dt = \int (kt + C)\,dt$$

$$= \frac{kt^2}{2} + Ct + D$$

We now use the information we have to determine the constant k, C and D:

$y(0) = 20$: then $20 = \frac{k(0)^2}{2} + C(0) + D$

$$D = 20$$

$y'(0) = 0$: then $0 = k(0) + C$

$$C = 0$$

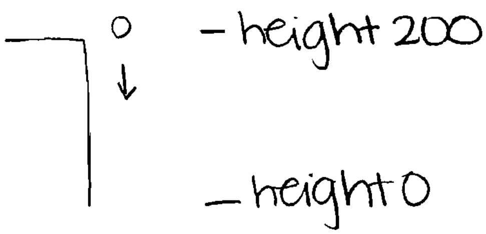
$y(2) = 0$: then $0 = \frac{k(2)^2}{2} + 20$

$$-20 = 2k$$

$$k = -10$$

Hence the acceleration due to gravity on this planet is -10 ft/sec^2.

Now we set up the second part of the problem:



$y(0)=200$
$y'(0)=0$

Now we know that the ball will hit the ground at some point, we just don't know when. Let's call that time $t=T$. Then

$$y(T)=0$$

We must now solve the DE $y''(t)=-10$. This has solution (by part 1)

$$y(t)=-5t^2+Ct+D$$

(C and D are different from part 1 because we have different initial values to our DE).

$y(0)=200$: then $200=-5(0)^2+C(0)+D$

$$D=200$$

$y'(0)=0$: then $0=-10(0)+C$

$$C=0$$

$y(T) = 0$: then $0 = -5T^2 + 200$

$$0 = -5(T^2 - 40)$$

so $T = 2\sqrt{10}$ or $T = -2\sqrt{10}$

obviously the ball did not hit the ground before it was thrown, so $T > 0$ and we choose $T = 2\sqrt{10}$.

So the ball takes $2\sqrt{10} \approx 6.32$ seconds to hit the ground and does so at a speed

$$|y'(2\sqrt{10})| = |-10(2\sqrt{10})| = 20\sqrt{10}$$

(Remember y' is the velocity and the speed is the absolute value of the velocity!)

which is about 63.25 ft/sec.

Section 1.3

\# 32. The differential equation is

$$y'(x) = 4x\sqrt{y}$$

and we must check that

$$y(x) = \begin{cases} 0 & \text{if } x^2 \le c \\ (x^2-c)^2 & \text{if } x^2 > c \end{cases}$$

for $c > 0$ satisfies the DE for all x.

If x is a real number, we can have:

① $x^2 < c$ i.e. $|x| < \sqrt{c}$ or $-\sqrt{c} < x < \sqrt{c}$
② $x^2 = c$ i.e. $x = \sqrt{c}$ or $x = -\sqrt{c}$
③ $x^2 > c$ i.e. $|x| > \sqrt{c}$ or $x > \sqrt{c}$ and $x < -\sqrt{c}$

We will check that y as defined above satisfies the DE on each of these 3 regions separately:

① if $|x| < \sqrt{c}$, then y is 0 everywhere in a neighborhood of x. Thus we may use derivatives rules to compute $y'(x) = 0$. Now if $|x| < \sqrt{c}$, then $y(x) = 0$, so $4x\sqrt{y} = 4x\sqrt{0} = 0$. Thus $y'(x) = 4x\sqrt{y}$ on this region.

② if $x = \sqrt{c}$ or $-\sqrt{c}$, then y is defined one way on the left of x and another way on the right of x. This is why we cannot use derivatives rules to compute $y'(x)$. We must go back to the definition of the derivative.

Reminder: a function $y(x)$ is said to be differentiable at x_0 if there is a number L such that

$$L = \lim_{h \to 0^-} \frac{y(x_0+h)-y(x_0)}{h} = \lim_{h \to 0^+} \frac{y(x_0+h)-y(x_0)}{h}$$

In this case we say $y'(x_0) = L$

So let's do it:

$x = \sqrt{c}$: on the left of $\sqrt{c}$, y is 0 and on the right of $\sqrt{c}$, y is $(x^2-c)^2$.

Define the functions $f(x) = 0$ for all x and $g(x) = (x^2-c)^2$ for all x. These functions are nice (they are polynomials) so we know they are differentiable everywhere, and in particular at $x = \sqrt{c}$. We also know that we can use derivatives rules on them, so

$$f'(\sqrt{c}) = 0$$

$$g'(\sqrt{c}) = 2((\sqrt{c})^2 - c) \cdot 2(\sqrt{c}) = 0$$

This tells us (using the definition of the derivative above) that

$$0 = \lim_{h \to 0^-} \frac{f(\sqrt{c}+h) - f(\sqrt{c})}{h}$$

and

$$0 = \lim_{h \to 0^+} \frac{g(\sqrt{c}+h) - g(\sqrt{c})}{h}$$

We may now investigate y.

$$\lim_{h \to 0^-} \frac{y(\sqrt{c}+h) - y(\sqrt{c})}{h} = \lim_{h \to 0^-} \frac{f(\sqrt{c}+h) - f(\sqrt{c})}{h} = 0$$

Since $h \to 0^-$, $\sqrt{c}+h$ is a little bit to the left of $\sqrt{c}$, and to the left of $\sqrt{c}$ we have $y=f$, so $y(\sqrt{c}+h) = f(\sqrt{c}+h)$. Also $y(\sqrt{c}) = f(\sqrt{c}) = 0$.

$$\lim_{h \to 0^+} \frac{y(\sqrt{c}+h) - y(\sqrt{c})}{h} = \lim_{h \to 0^+} \frac{g(\sqrt{c}+h) - g(\sqrt{c})}{h} = 0$$

Since $h \to 0^+$, $\sqrt{c}+h$ is a little bit to the right of $\sqrt{c}$, and there we have $y=g$, so $y(\sqrt{c}+h) = g(\sqrt{c}+h)$. Also $y(\sqrt{c}) = g(\sqrt{c}) = 0$.

Thus $y'(\sqrt{c}) = 0$. We have that $4\sqrt{c}\sqrt{0} = 0$, so the DE is satisfied at $\sqrt{c}$.

$x = -\sqrt{c}$: On the left of $-\sqrt{c}$, y is $(x^2-c)^2$ and on the right of $-\sqrt{c}$, y is 0. Thus keeping f and g defined as above we have the following facts:

- $y(-\sqrt{c}) = g(-\sqrt{c}) = f(-\sqrt{c}) = 0$
- g and f are both differentiable and
  $g'(-\sqrt{c}) = 2((-\sqrt{c})^2 - c)\cdot 2(-\sqrt{c}) = 0$
  $f'(-\sqrt{c}) = 0$
- for h small and negative,
  $$y(-\sqrt{c}+h) = g(-\sqrt{c}+h)$$
- for h small and positive,
  $$y(-\sqrt{c}+h) = f(-\sqrt{c}+h)$$

and from these facts we get

$$\lim_{h\to 0^-} \frac{y(-\sqrt{c}+h) - y(-\sqrt{c})}{h} = \lim_{h\to 0^-} \frac{g(-\sqrt{c}+h) - g(-\sqrt{c})}{h} = 0$$

$$\lim_{h\to 0^+} \frac{y(-\sqrt{c}+h) - y(-\sqrt{c})}{h} = \lim_{h\to 0^-} \frac{f(-\sqrt{c}+h) - f(-\sqrt{c})}{h} = 0$$

so $y'(-\sqrt{c}) = 0$. We have that $4(-\sqrt{c})(\sqrt{0}) = 0$, so the DE is satisfied at $-\sqrt{c}$.

③ if $|x| > \sqrt{c}$, then y is $(x^2-c)^2$ everywhere on a neighborhood around x and we may use derivatives rules again.

$$y'(x) = 2(x^2-c)\cdot 2x = 4x(x^2-c)$$

$$4x\sqrt{y} = 4x\sqrt{(x^2-c)^2} = 4x\,|x^2-c|$$

since $|x| > \sqrt{c}$, x^2-c is positive so $|x^2-c| = x^2-c$ and the DE is satisfied.

I will not finish this problem.