The Lights Out Problem

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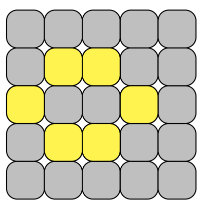
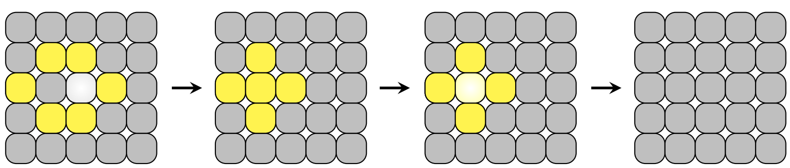
Lights Out is an electronic game from 1995 consisting of a 5 by 5 grid of buttons which light up. At the beginning of the game, a random collection of the buttons are lit up. The goal, indicated by the name of the game, is to turn off all the lights. Pushing a button changes the state of that light from off to on or on to off. But there is a catch: pushing a button also changes the state of the adjacent lights, those which share a side with the button you pushed.

For example, suppose we were given the following starting configuration,

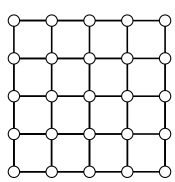


where the yellow squares indicate lights which are turned on, and the grey squares indicate lights which are turned off. It may not be obvious, but we can win in two steps this round by pressing the white buttons below.



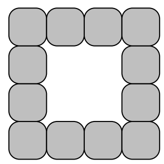
In order to understand this problem better, we can turn the game board into a kind of graph, called a *grid graph.* This may prove illuminating. At very least, we will have the wealth of known information about graph theory to work with.

If we make each button into a *node*, or *vertex*, and connect buttons which share an edge, we obtain the following graph.



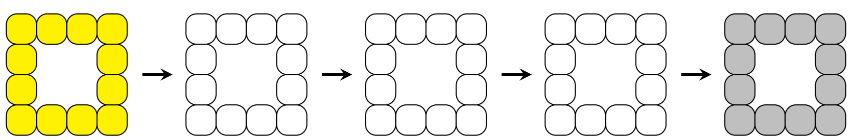
We can color the vertices either white or black corresponding to the states on or off and play the same game as before on this graph, just with a different picture. Pushing buttons now means changing the state of a vertex and each of its neighbors.

Problem 1. Imagine that you buy a mini version of *Lights Out*, but it comes missing four buttons in the middle of the board. Somehow the game still works, and you would like to study this one.

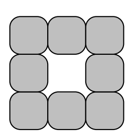


Draw the graph which would correspond to this game.

Problem 2. When you turn on the mini-Lights Out game from the last problem, all of the lights turn on. Can you find a solution to turn them all off in four steps? Color the blank grids to show the intermediate steps.



Problem 3. Does the same solution work for the following Lights Out game board? Why or why not?



Problem 4 (Challenge)**.** Can you think of a method that would turn all of the lights from on to off given *any* cycle of buttons? Can you prove that your method always works?

The following mathematical problem predates the electronic game by nearly 20 years. It was posed by Lázló Lovász in a book entitled *Cominatorial Problems and Exercises* [1].

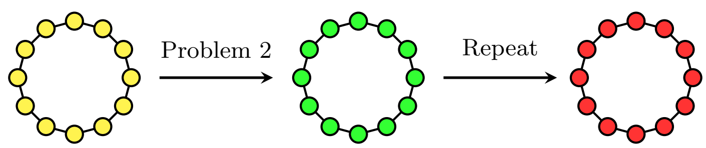
Problem 5 (Super challenge question)**.** Take any simple graph (collection of vertices with edges between some pairs of vertices). Imagine that at each vertex there is a light and a button. Pressing a button changes the state of the light at that vertex and the states of the lights at its neighbors. At the beginning all lights are on. Show that one can push some buttons so that all of the lights will be turned off.

More Colors!

Now imagine that, instead of alternating between two states, off and on, the lights now cycle through three states, say red, yellow, and green. The result of the Lovász' question is really quite amazing: given any graph on which to play Lights Out, if we start with all of the buttons (or vertices) lit up, then we can always turn off all of the lights in some finite number of steps.

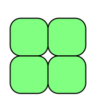
It feels natural to ask the same thing about traffic lights. That is, if we are given any graph to play Lights Out on and the lights cycle through the three states red, yellow, and green, if the lights are all green at the beginning, can we make them all red? Or can we make them all yellow?

This seems like a complicated question, so let's try some examples. If we take the graph corresponding to the mini-Lights Out game from Problems 1 and 2, it seems like the strategy from Problem 2 should do it! That is, since we changed the state of each light exactly one time in Problem 2, we can repeat the process as many times as we need to depending on the number of colors and the desired outcome. That is,



So, does this work for any graph? Lets examine a different example.

Problem 6. You have become so enticed by Lights Out that you buy yet another miniature version, but this one has lights which cycle from red to yellow to green to red. When you turn it on, the buttons all light up green, like this:



Is it possible to turn all the lights to red? Or to yellow? Why or why not?

Hint: Think about the number of lights which change color when a button is pressed and how many times each light would need to change state overall.

Problem 7 (Challenge)**.** Can you think of a condition that we could impose on our graph in order to ensure that we can turn the lights from all green to all red? Moreover, can you think of a condition so that a graph may be solved on any number of colors?

Problem 8 (Extra Challenge)**.** Suppose that now the lights cycle through c colors, $0, 1, \ldots, c-1$. We call a simple graph G *c-solvable* if we can start with all lights in state 1, press some buttons, and turn all of the lights to state 0 (whatever colors those numbers correspond to).

Show that, if each of the n vertices of a simple graph G has k neighbors and G is solvable by pressing t buttons, then

$$(k+1)t \equiv -n \pmod{c}.$$

Problem 9 (Open question)**.** Does the converse of the previous problem hold? That is, if a simple, k-regular graph G is such that $(k+1)t \equiv -n \pmod{c}$ for some positive integer t, must G be c-solvable?

References

[1] L Lovász. Combinatorial problems and exercises. 1979. pg. 42.