THE UNIVERSITY OF VERMONT
COLLEGE OF ENGINEERING &
MATHEMATICAL SCIENCES

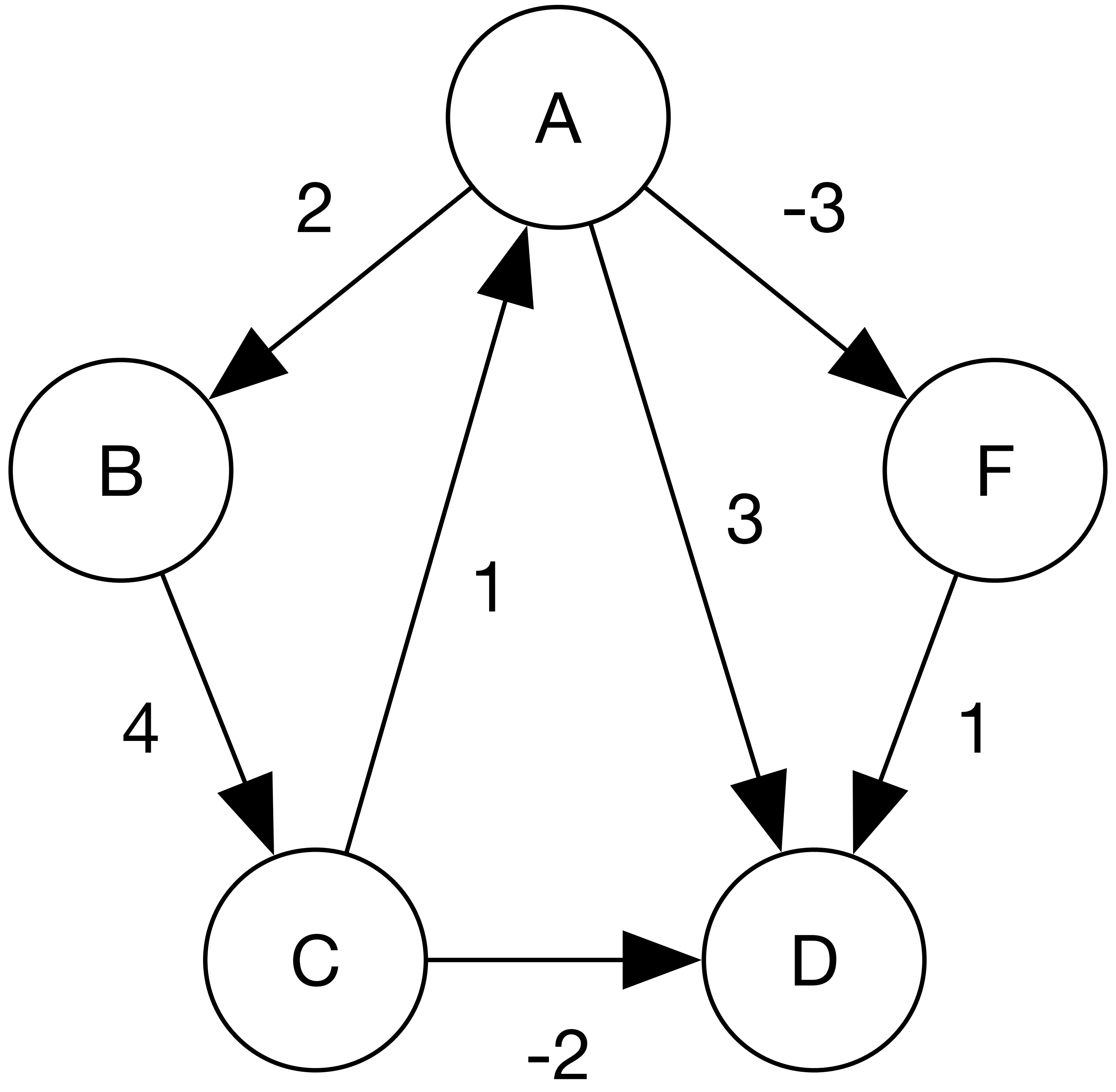
Shortest Path

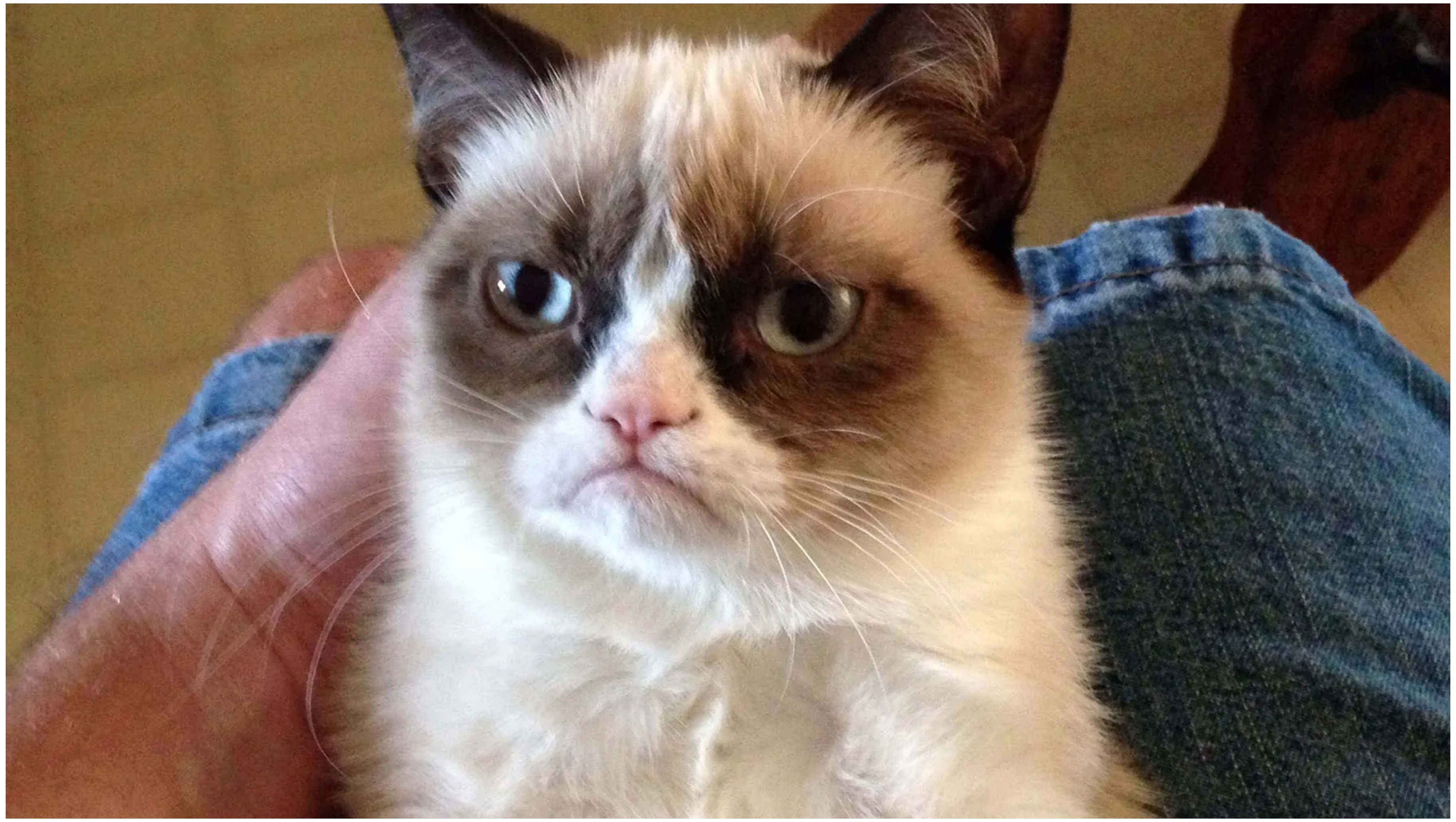
Bellman-Ford Algorithm

CS 124 / Department of Computer Science

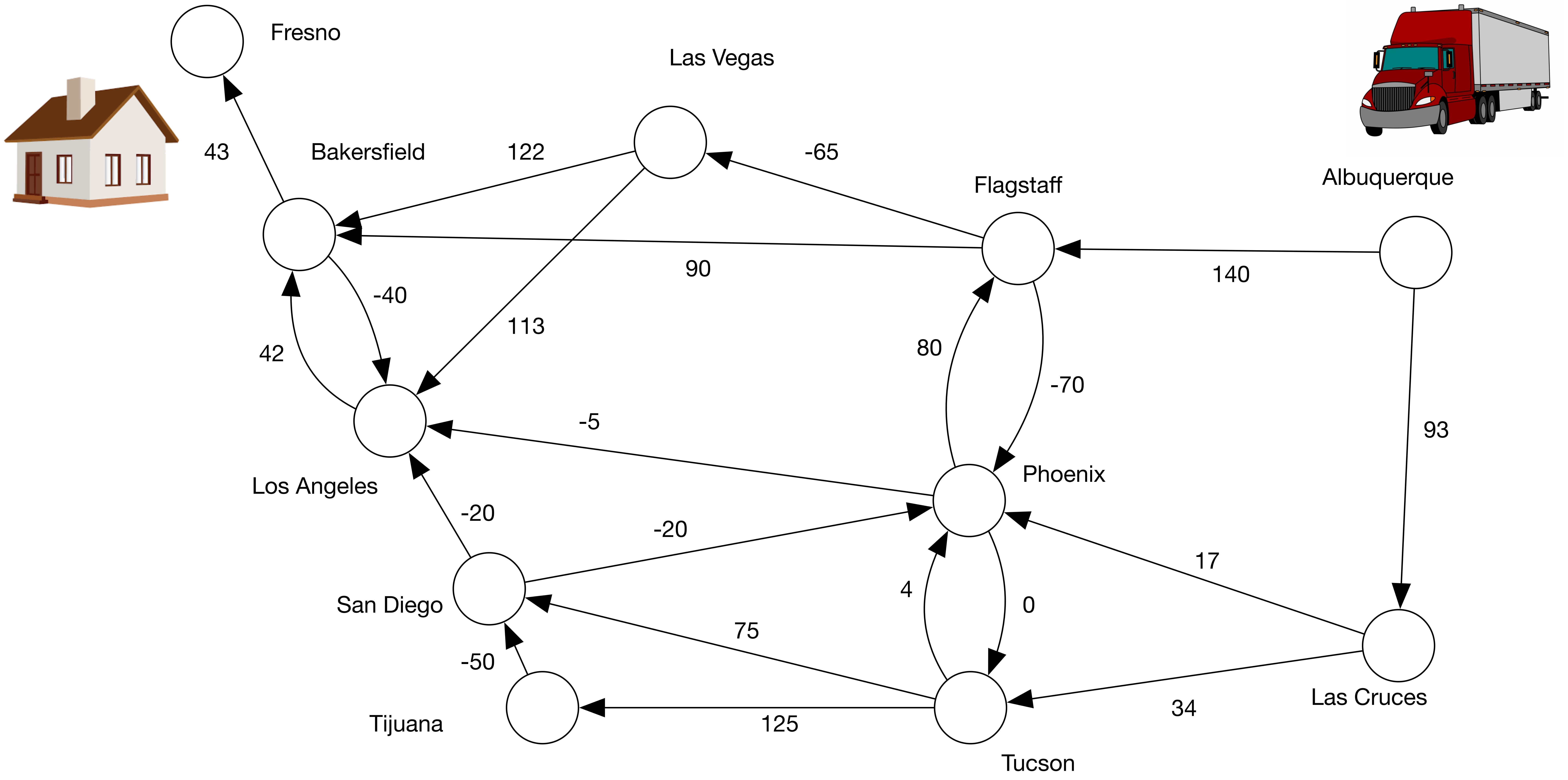
Shortest path

Dijkstra's algorithm doesn't work if there's an edge in the graph with negative weight.





Lowest-cost path with negative weights



What can be done?

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Bellman-Ford algorithm

Bellman-Ford

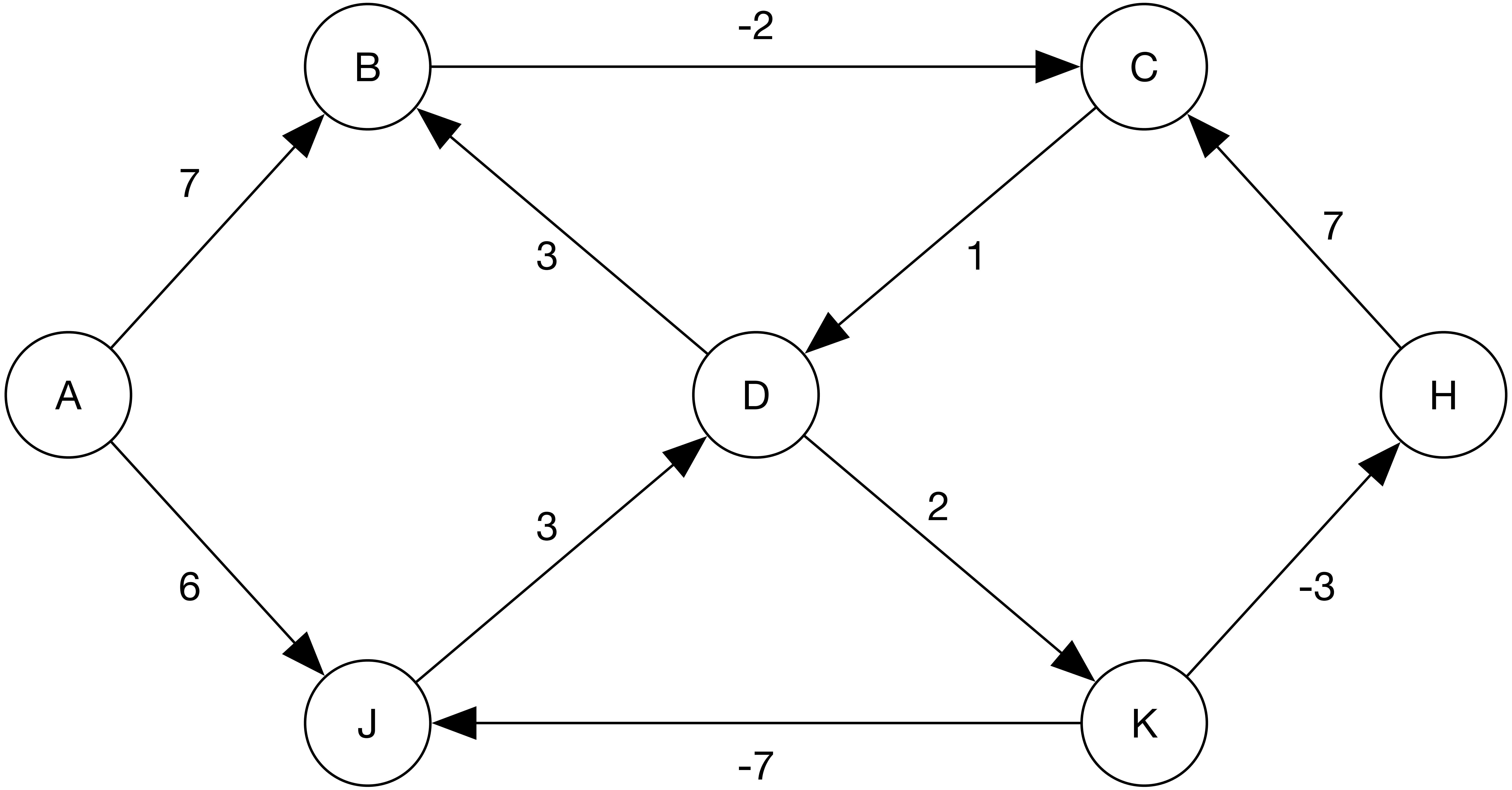
Given some graph, G = (V, E), and some starting node S ∈ V, the Bellman-Ford algorithm will find the shortest paths (or paths with minimum weight) from S to all other nodes in V.

Note that G must not contain any negative weight cycles.

Shortest Path / Lowest Cost Path

Algorithm	Dijkstra	Bellman - Ford
Worst-case complexity	$O((\lvert V\rvert + \lvert E\rvert) \log \lvert V\rvert)$	$O(\lvert V\rvert \times \lvert E\rvert)$
Restrictions	Edge weights must be non-negative	No negative cycles

What is a negative cycle?



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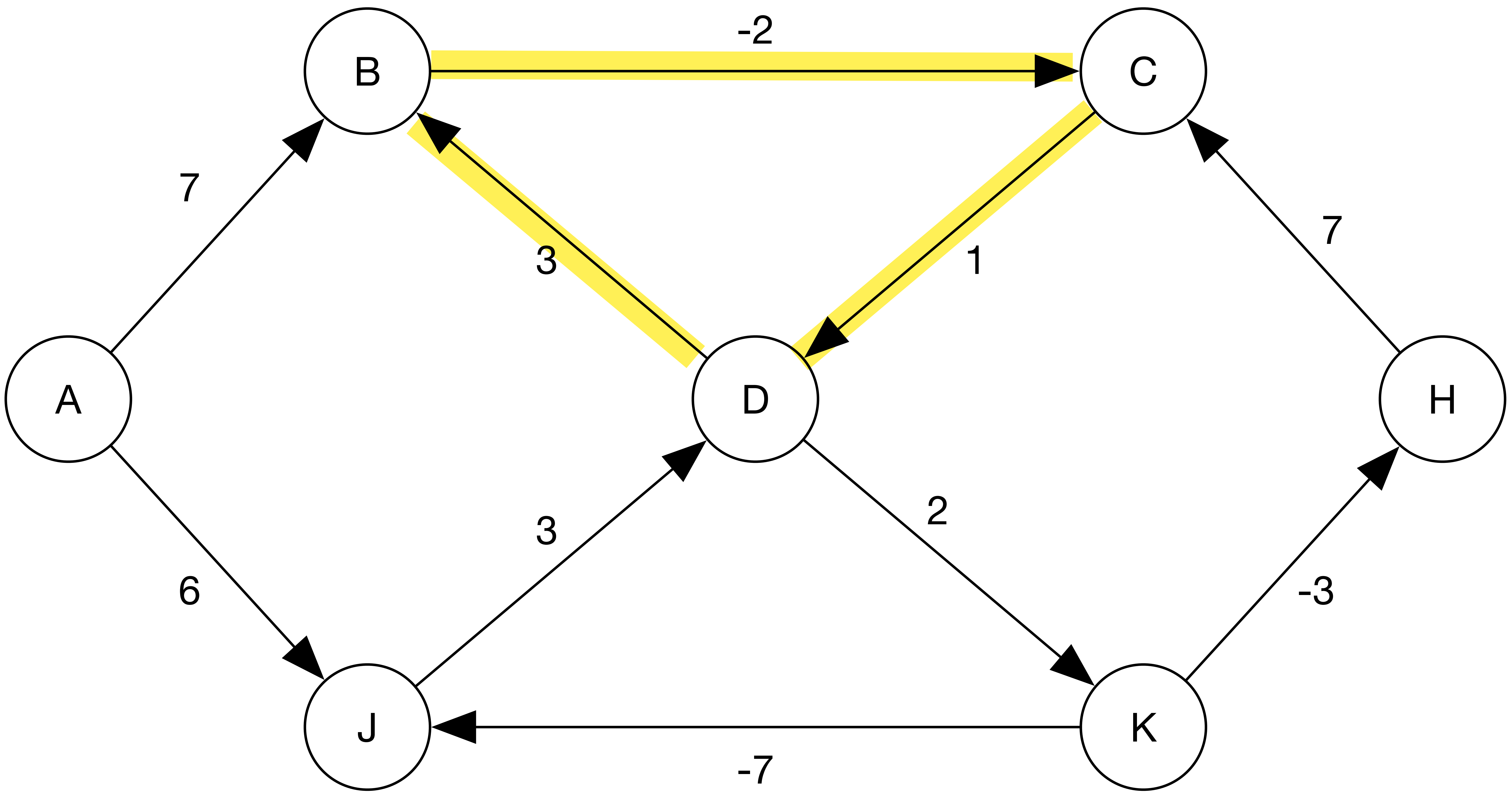

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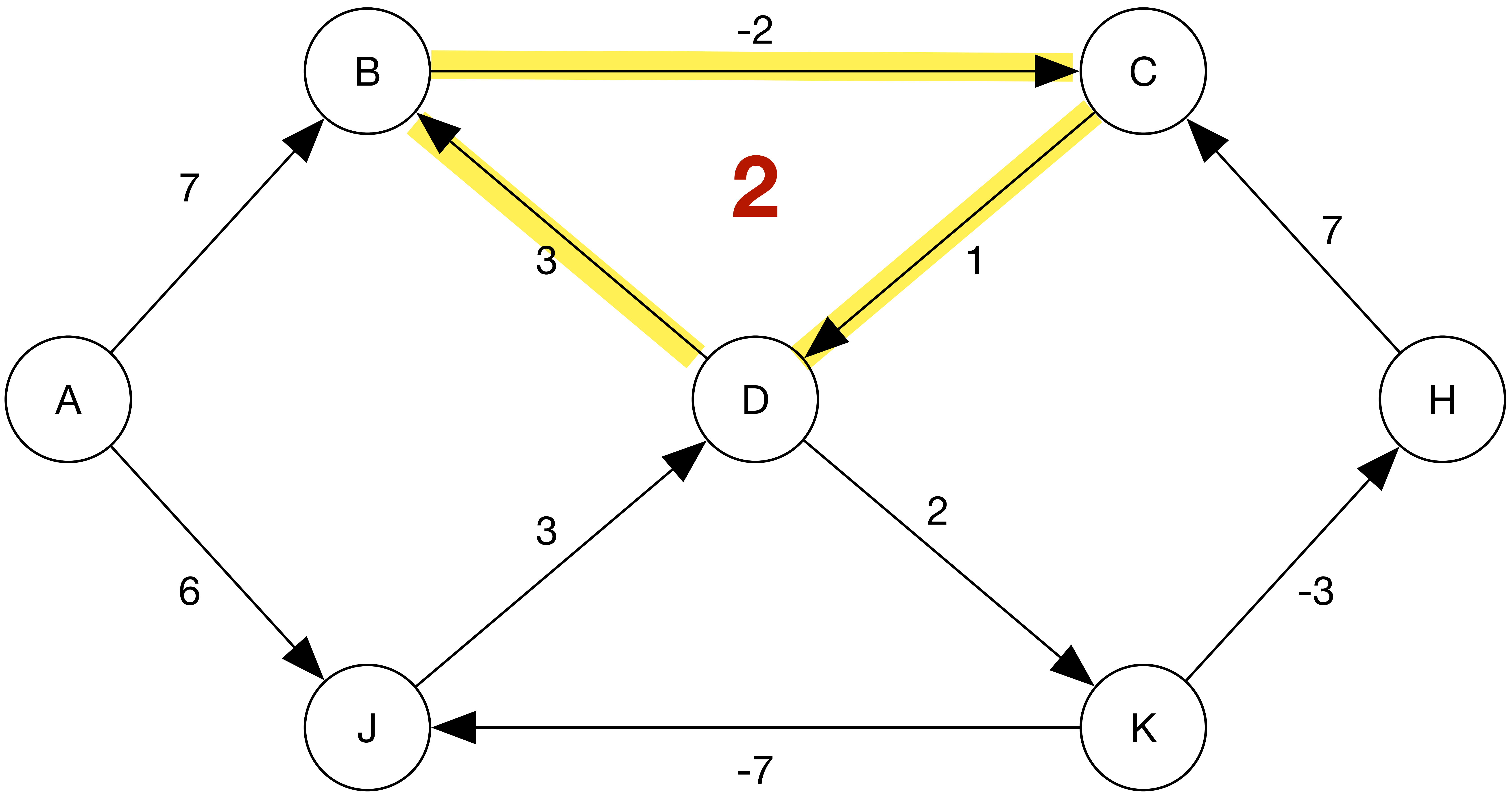

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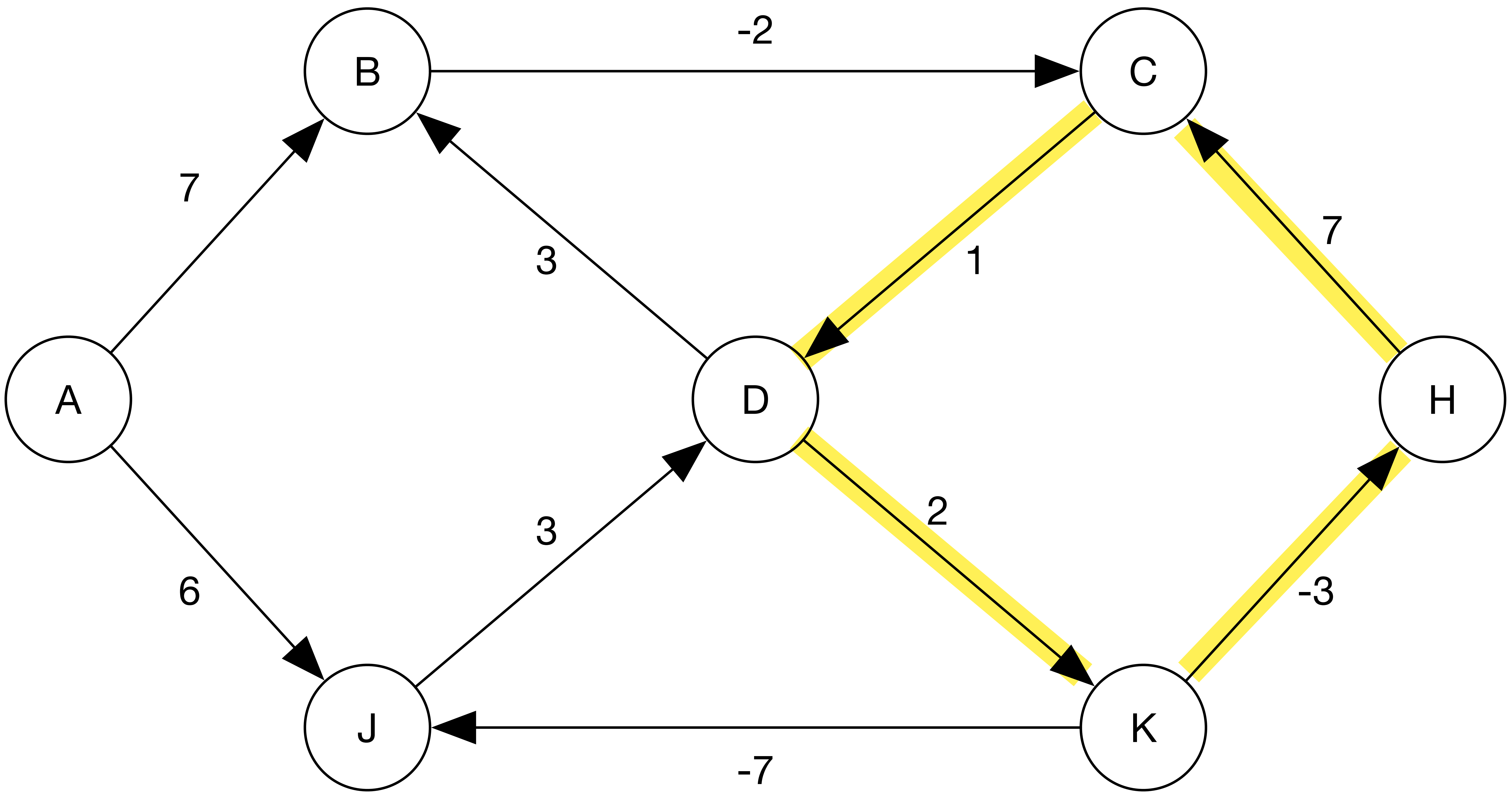

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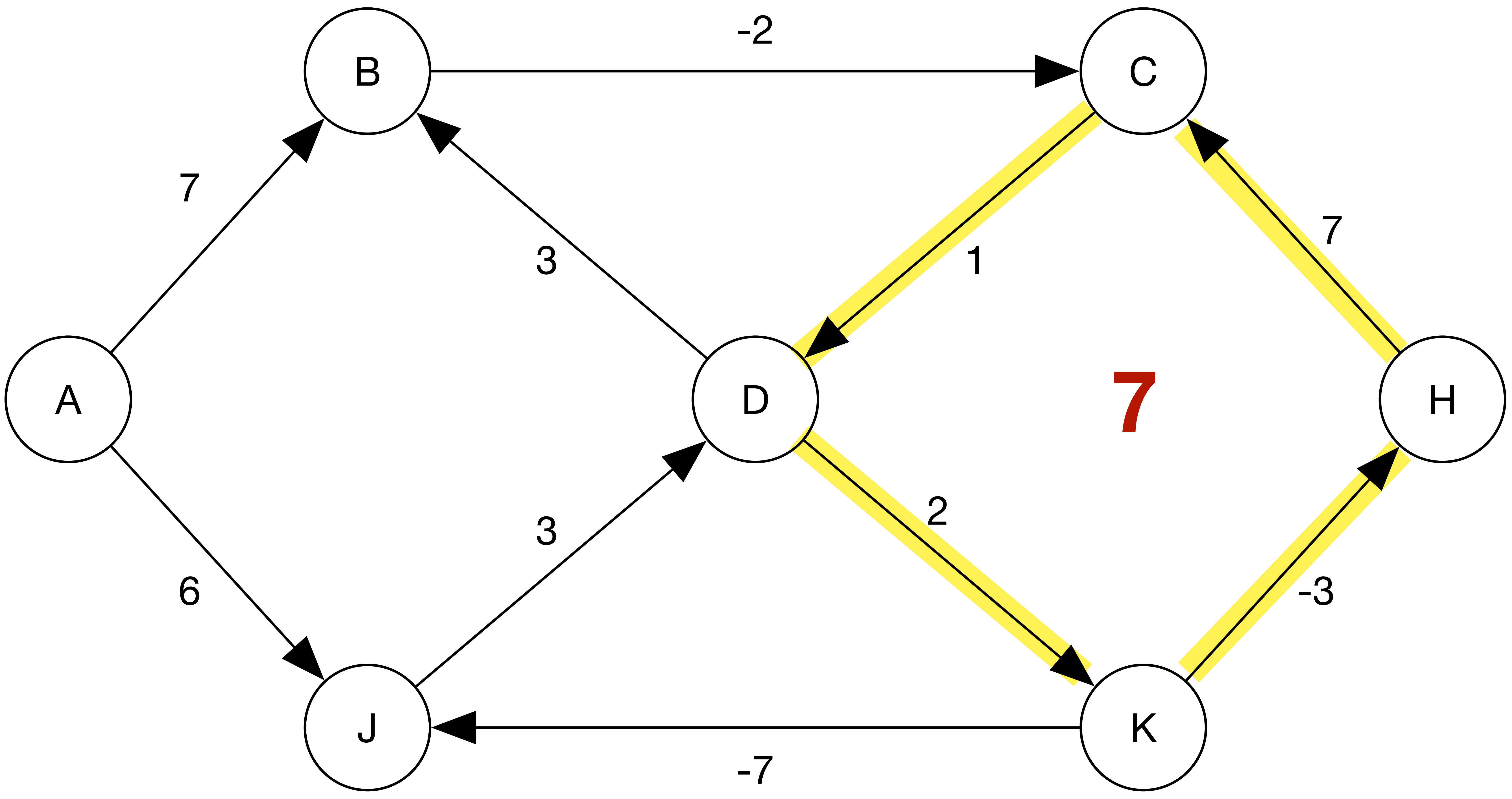

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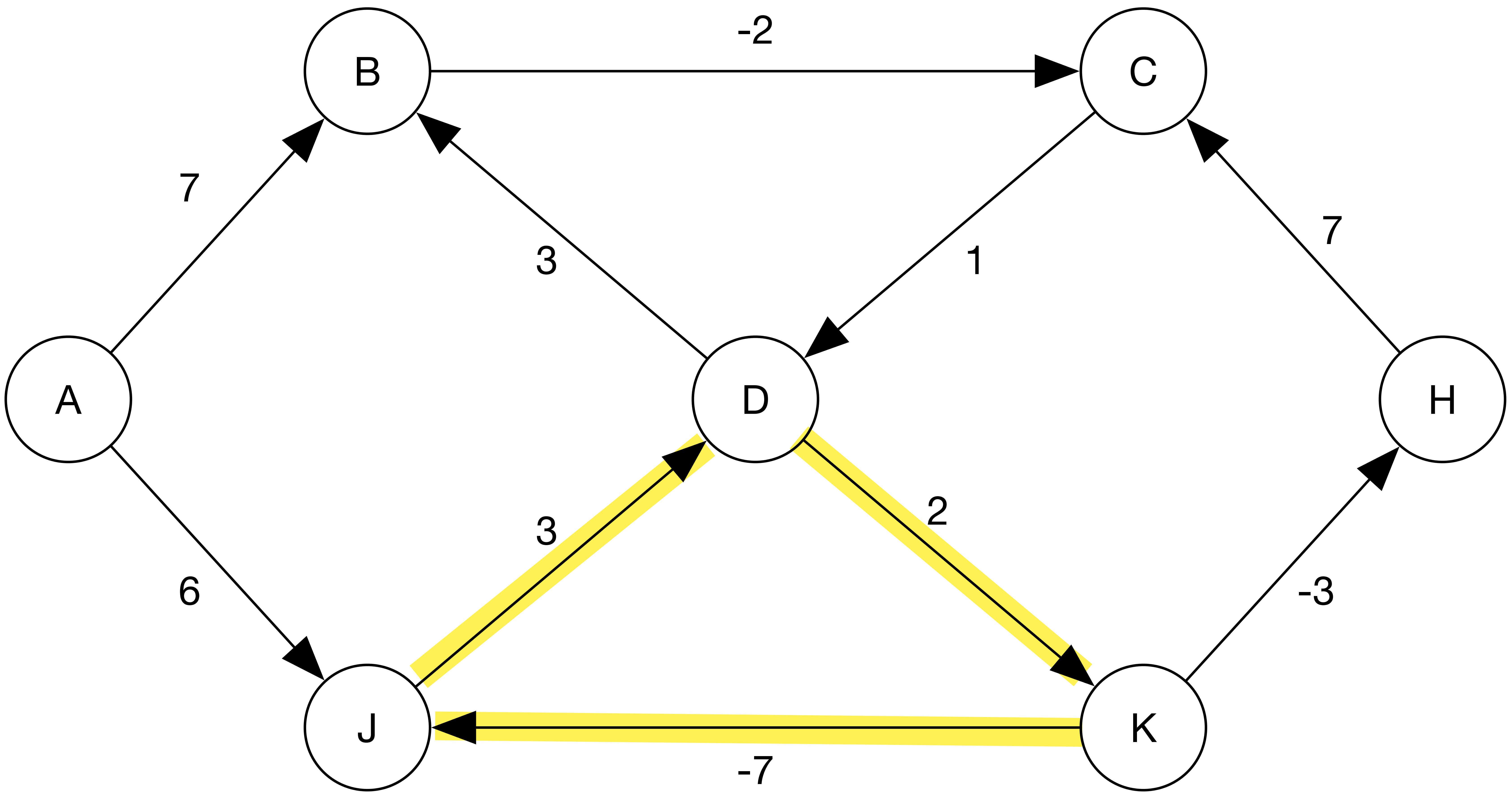

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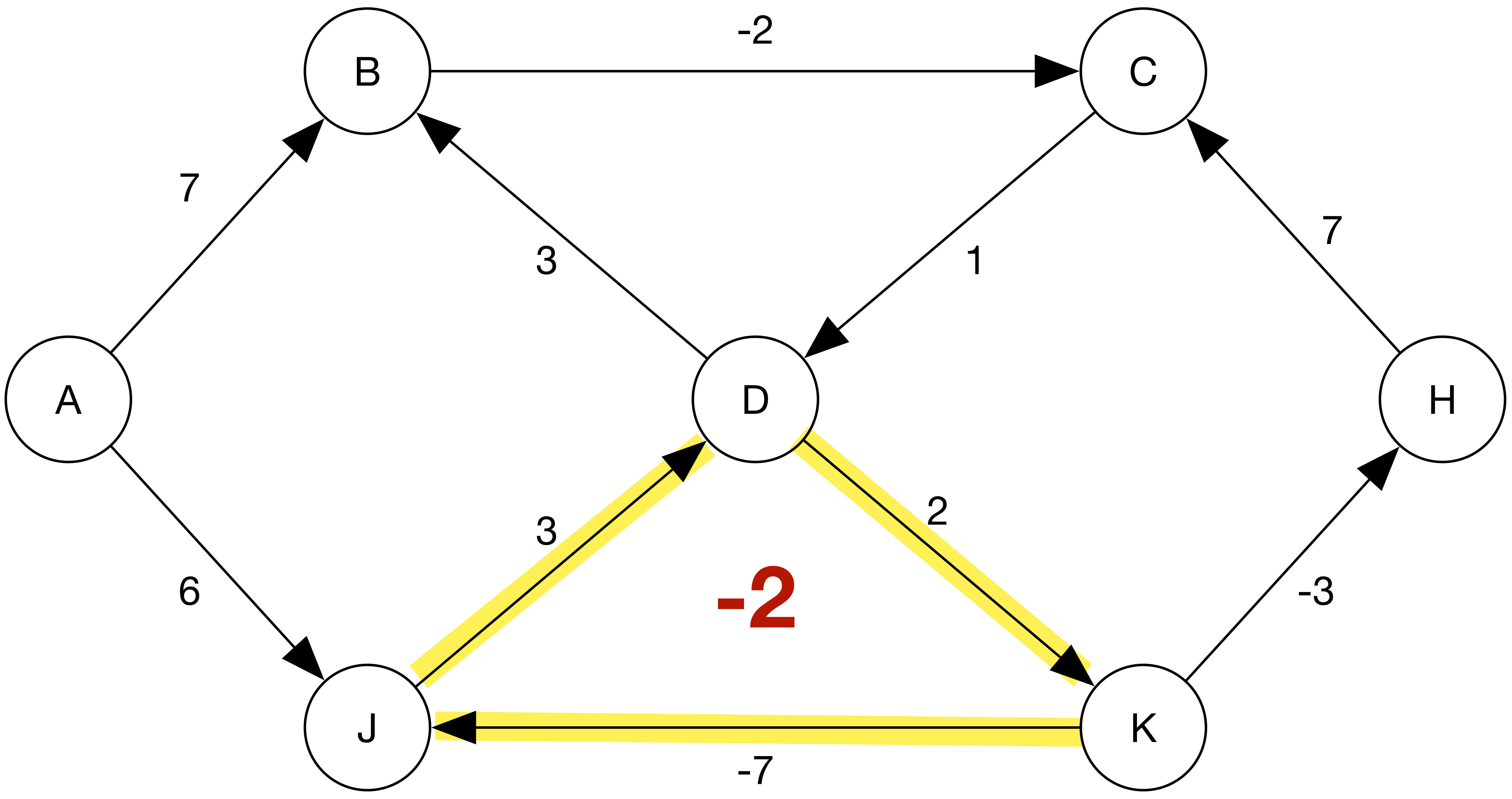

Bellman-Ford: pseudocode

```
function bellman_ford(G, S)  // G is the graph; S is the starting node
    for each node V in G
        arrived_from[V] = null
        if V = S
            distance[V] = 0
        else
            distance[V] = infinite

   repeat |V| - 1 times or until no distances are updated
        for each edge (U, V) in E
            distance = distance[U] + weight of edge
            if distance < distance[V]  // We've found a shorter distance
                distance[V] = distance
                arrived_from[V] = U

   for each edge (U, V) in E
       if distance[V] > distance[U] + weight of edge
           return "ERROR: negative weight cycle"
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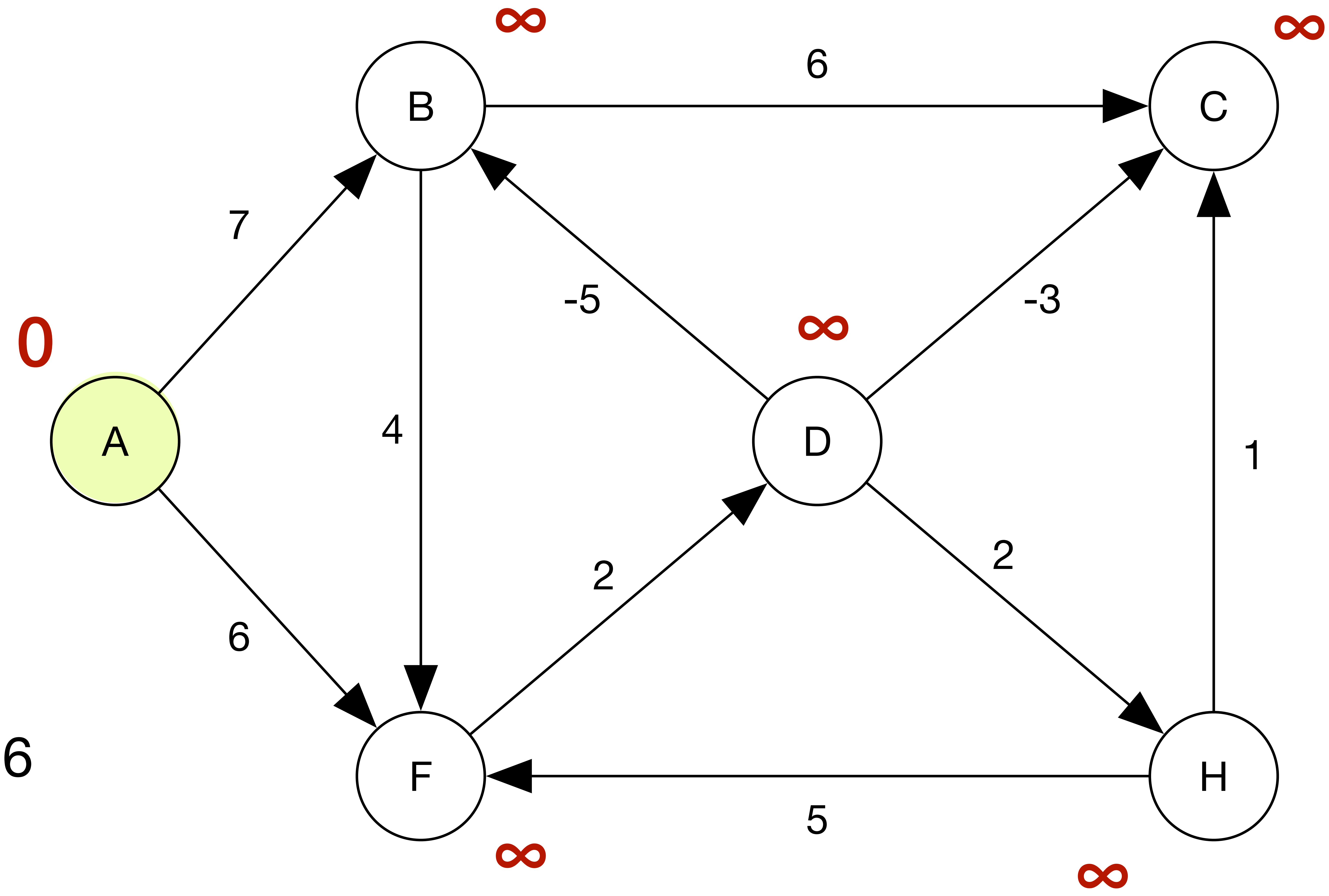

$|V| = 6$

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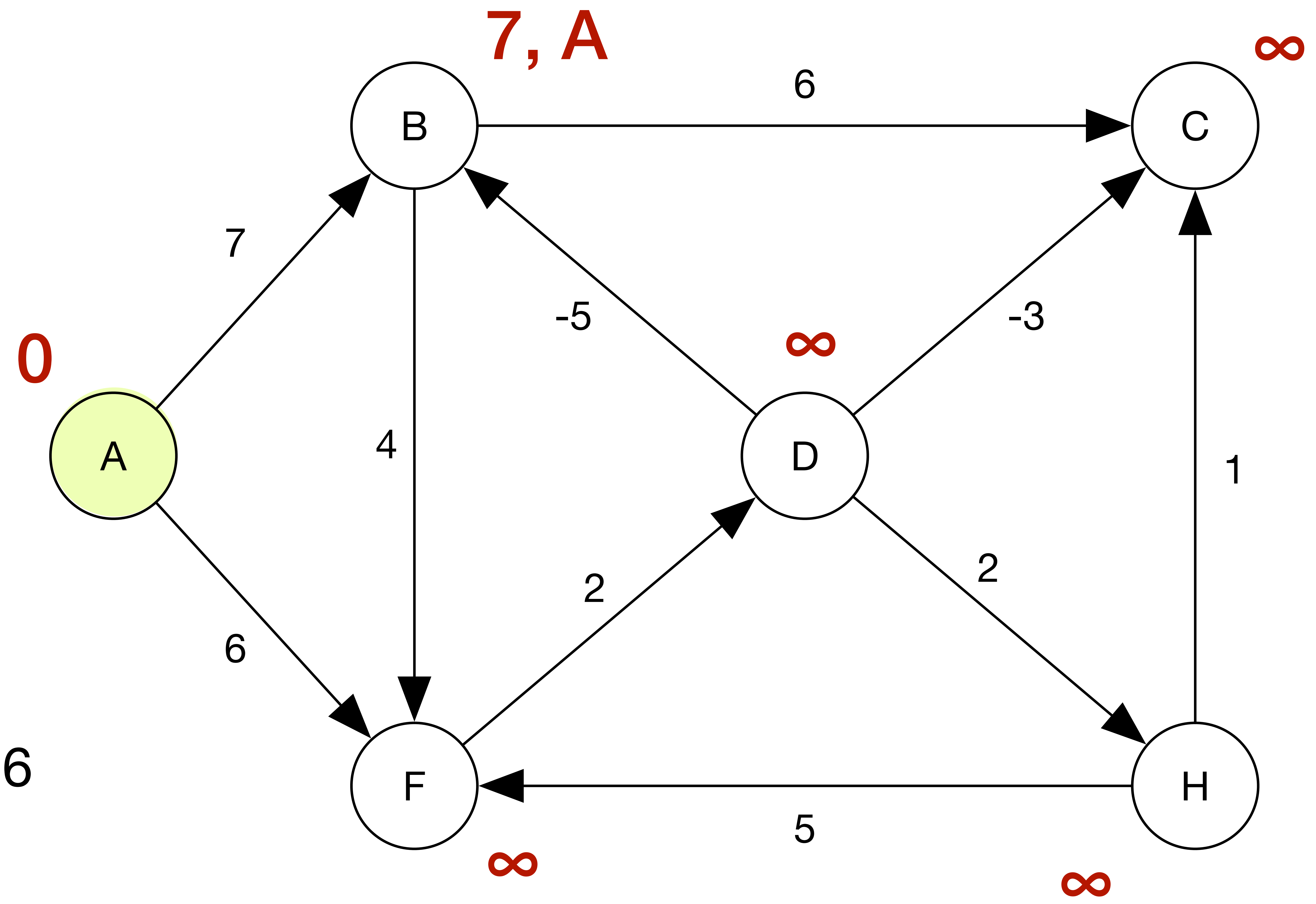
Bellman-Ford

First iteration



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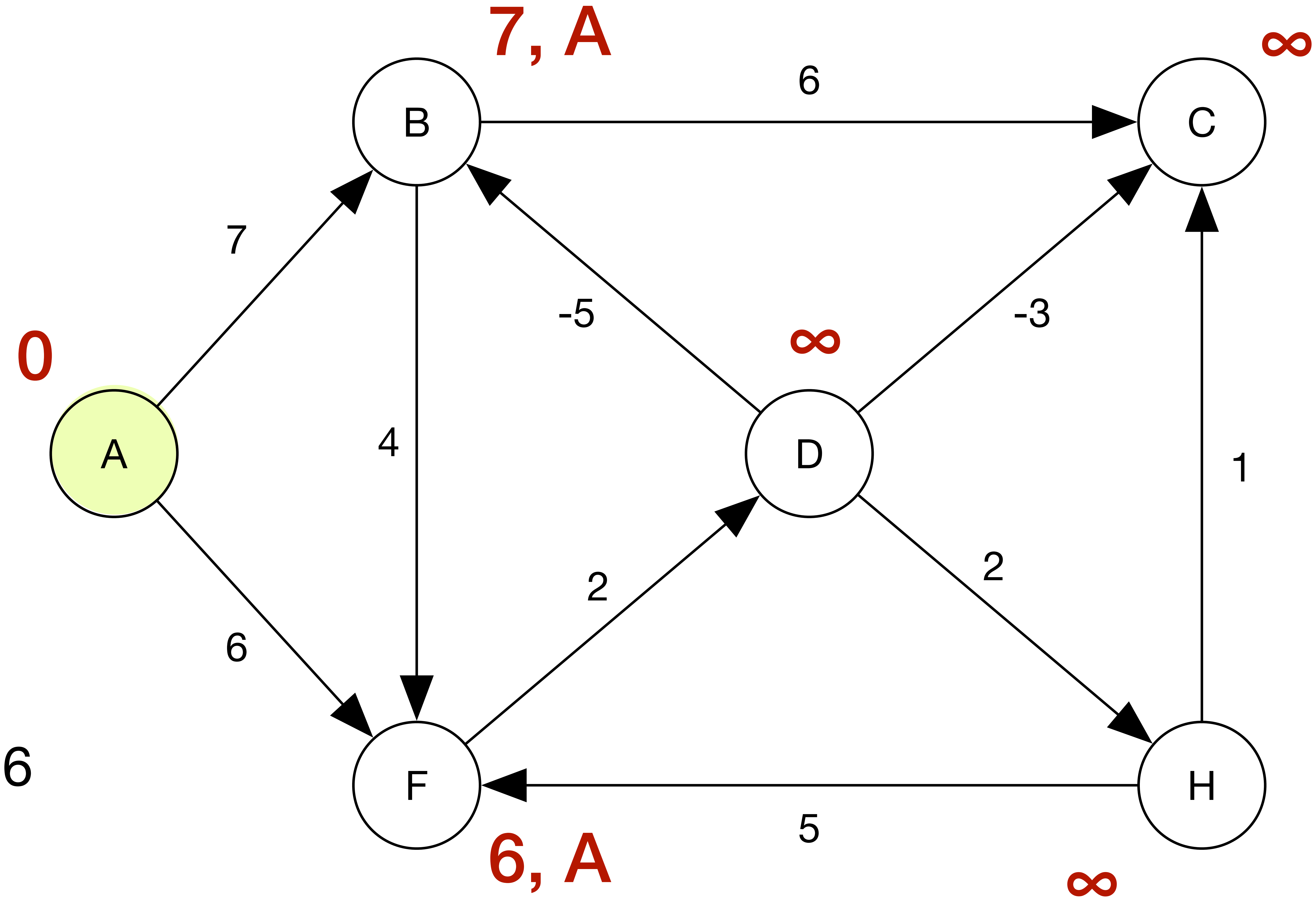

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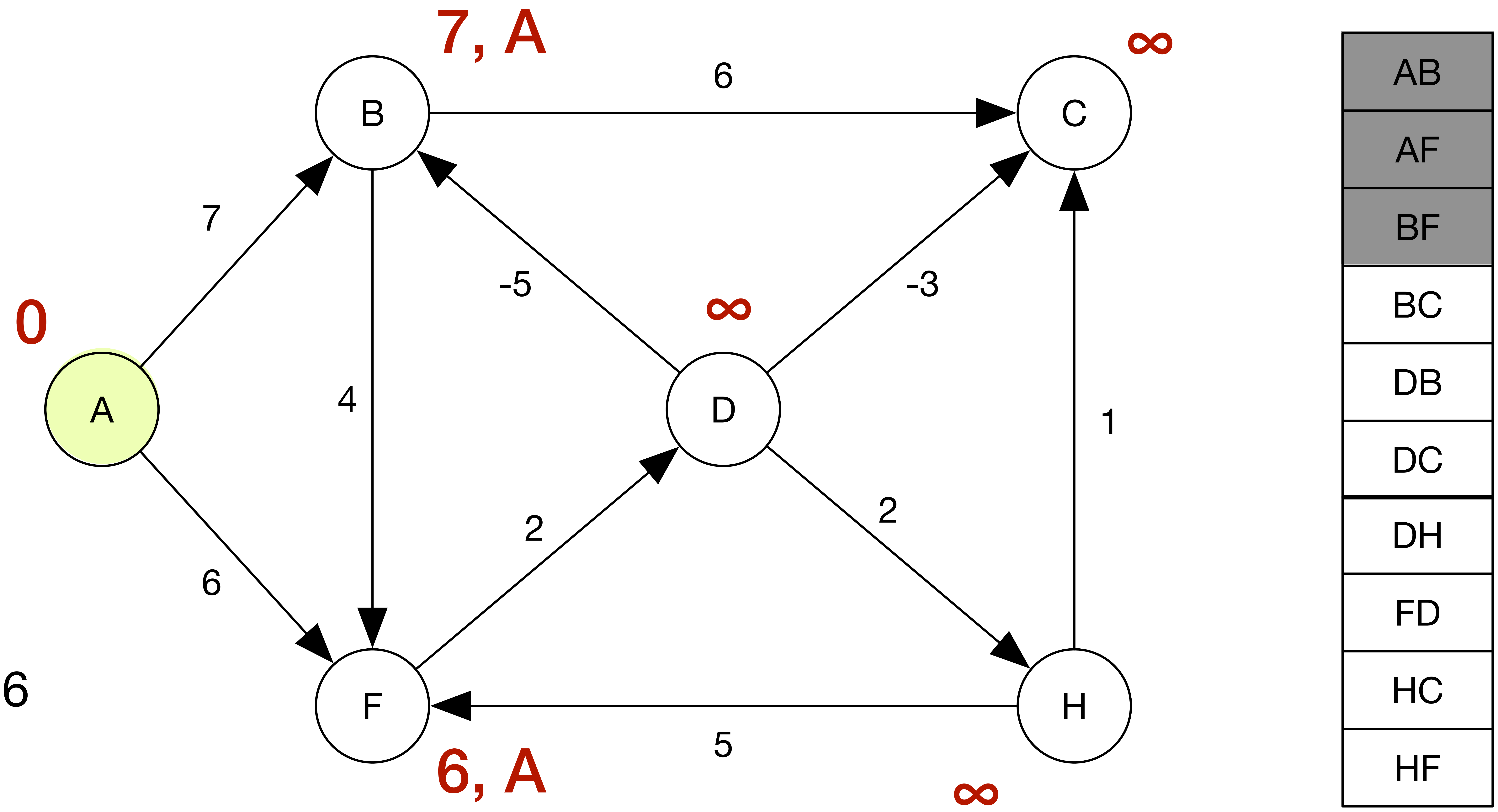

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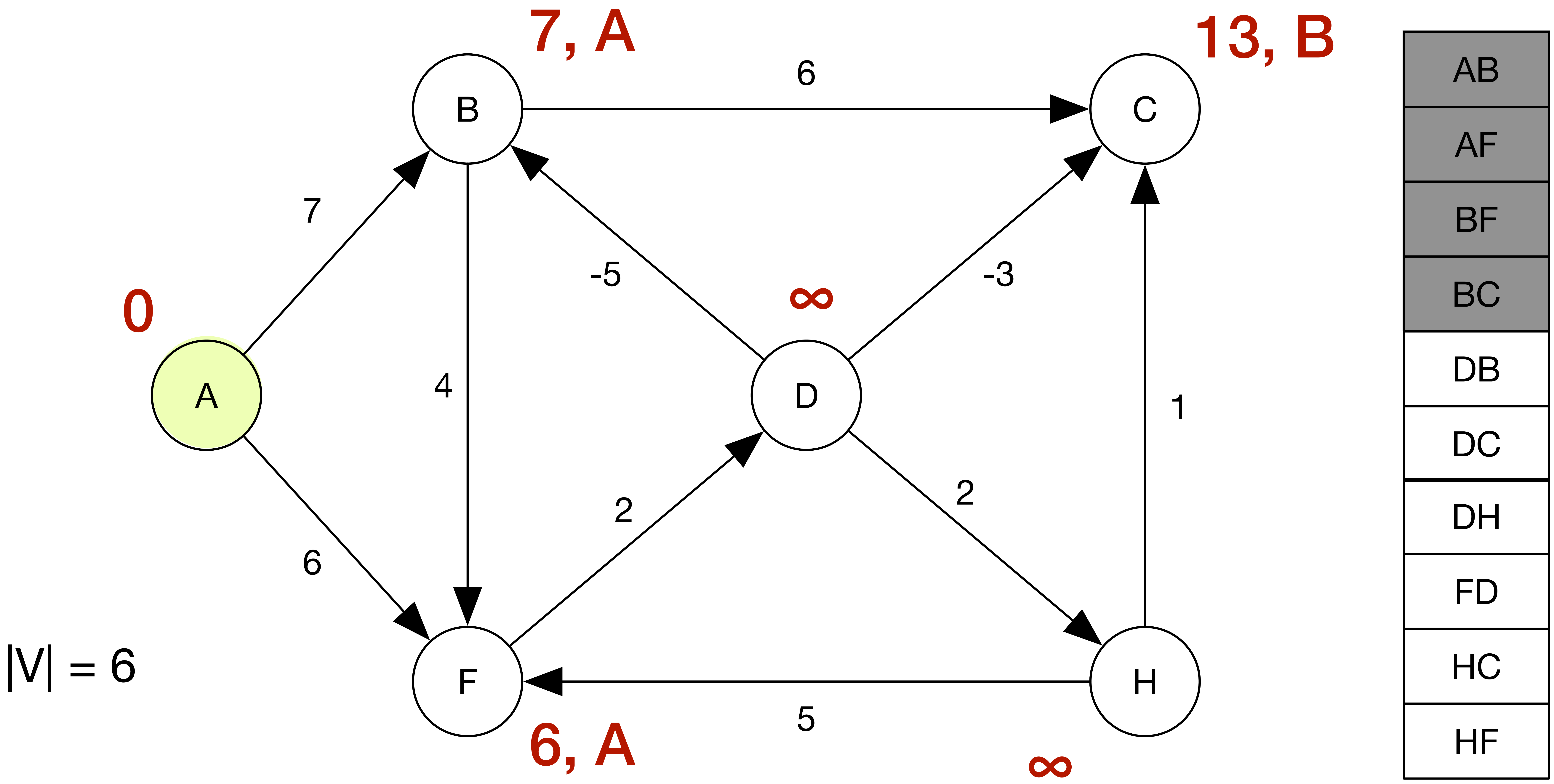

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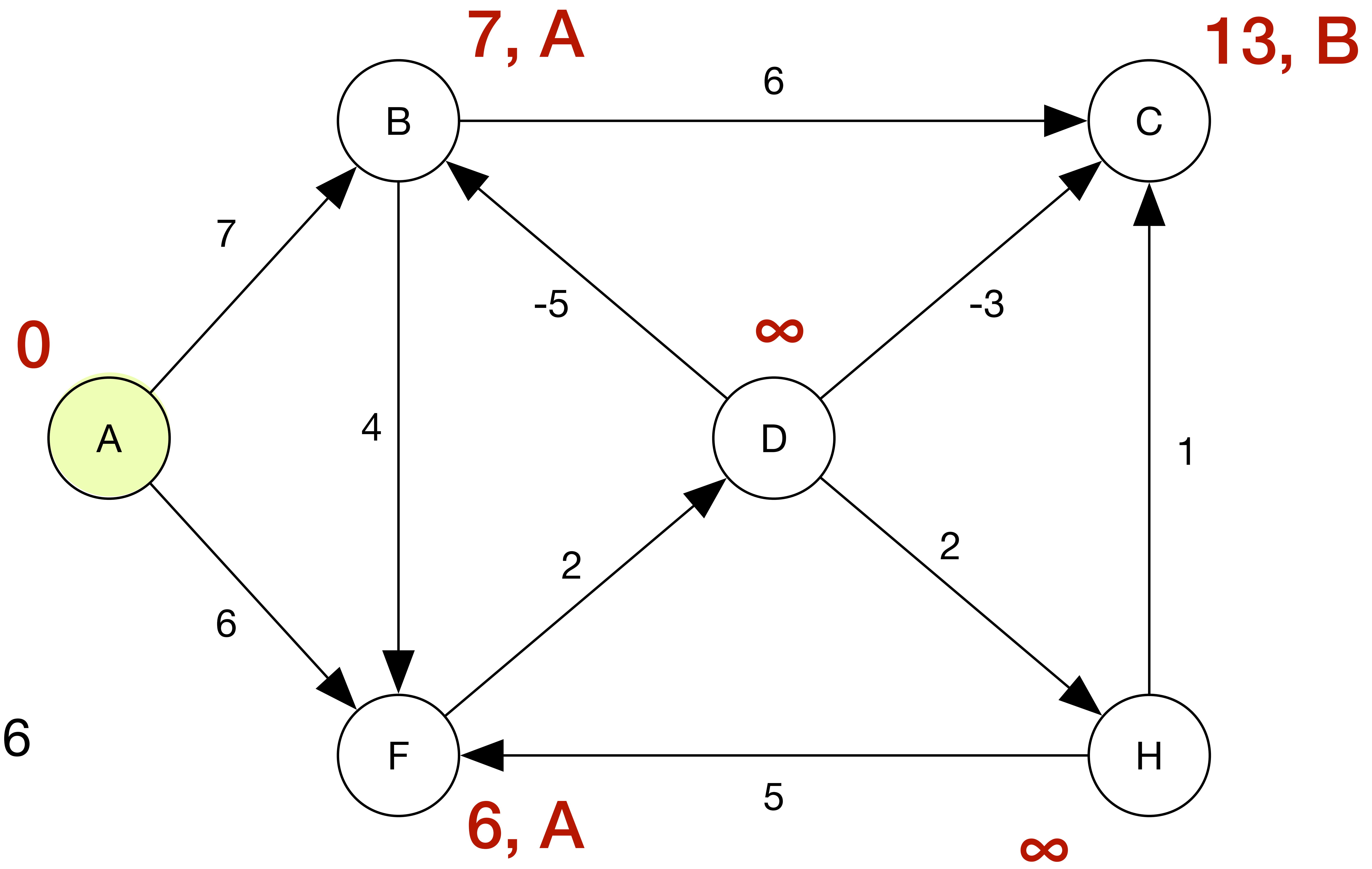

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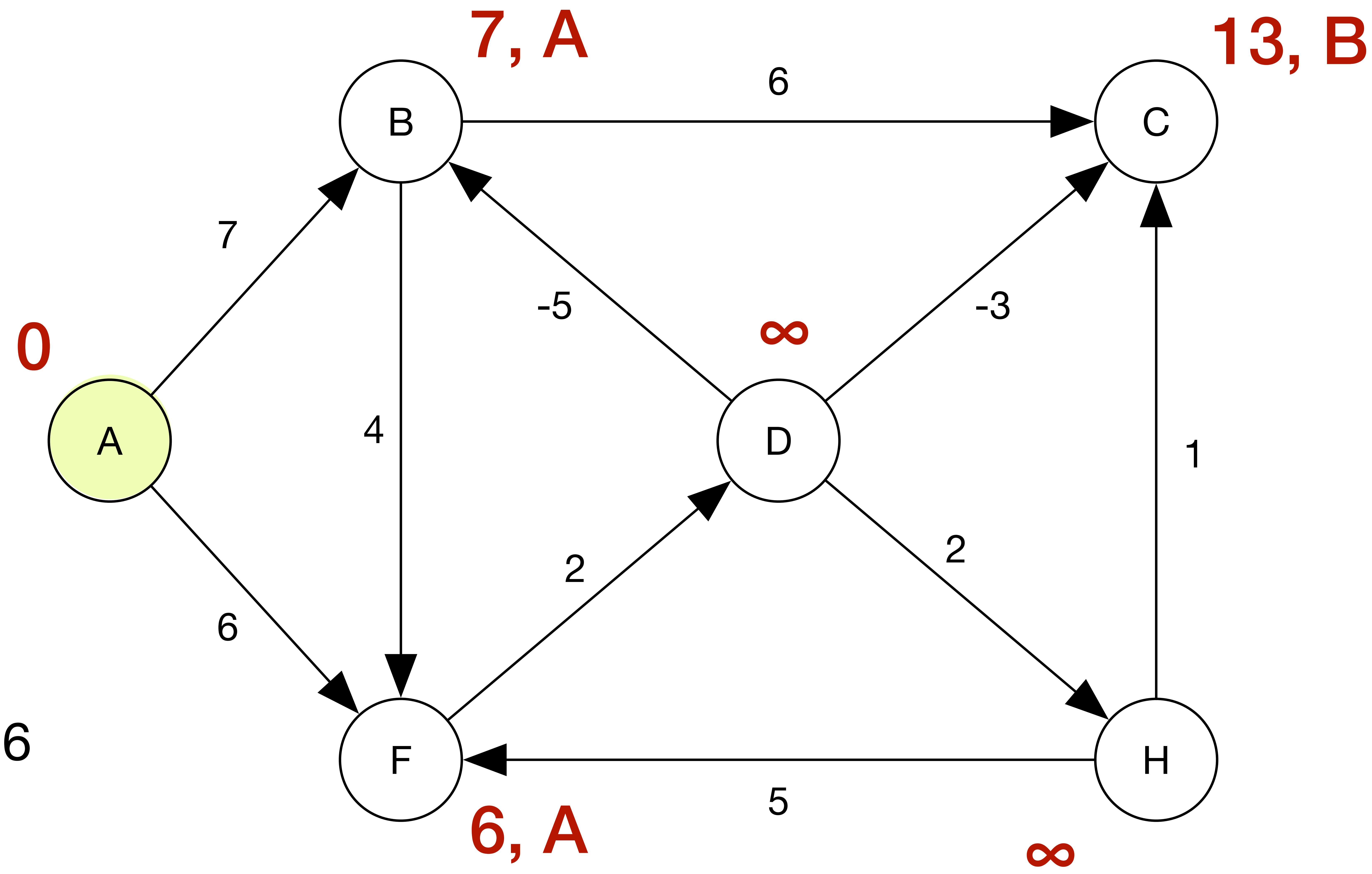

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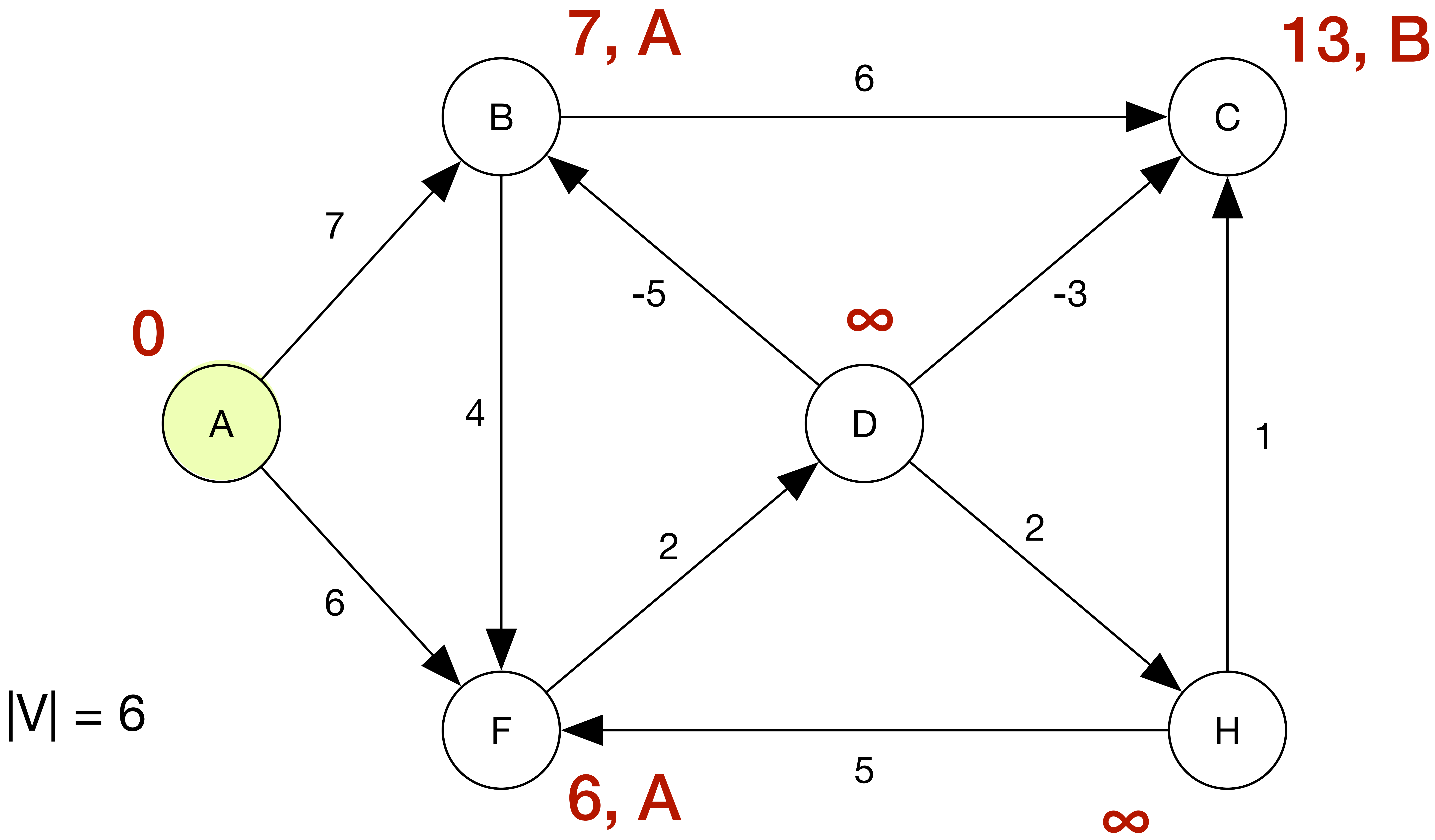

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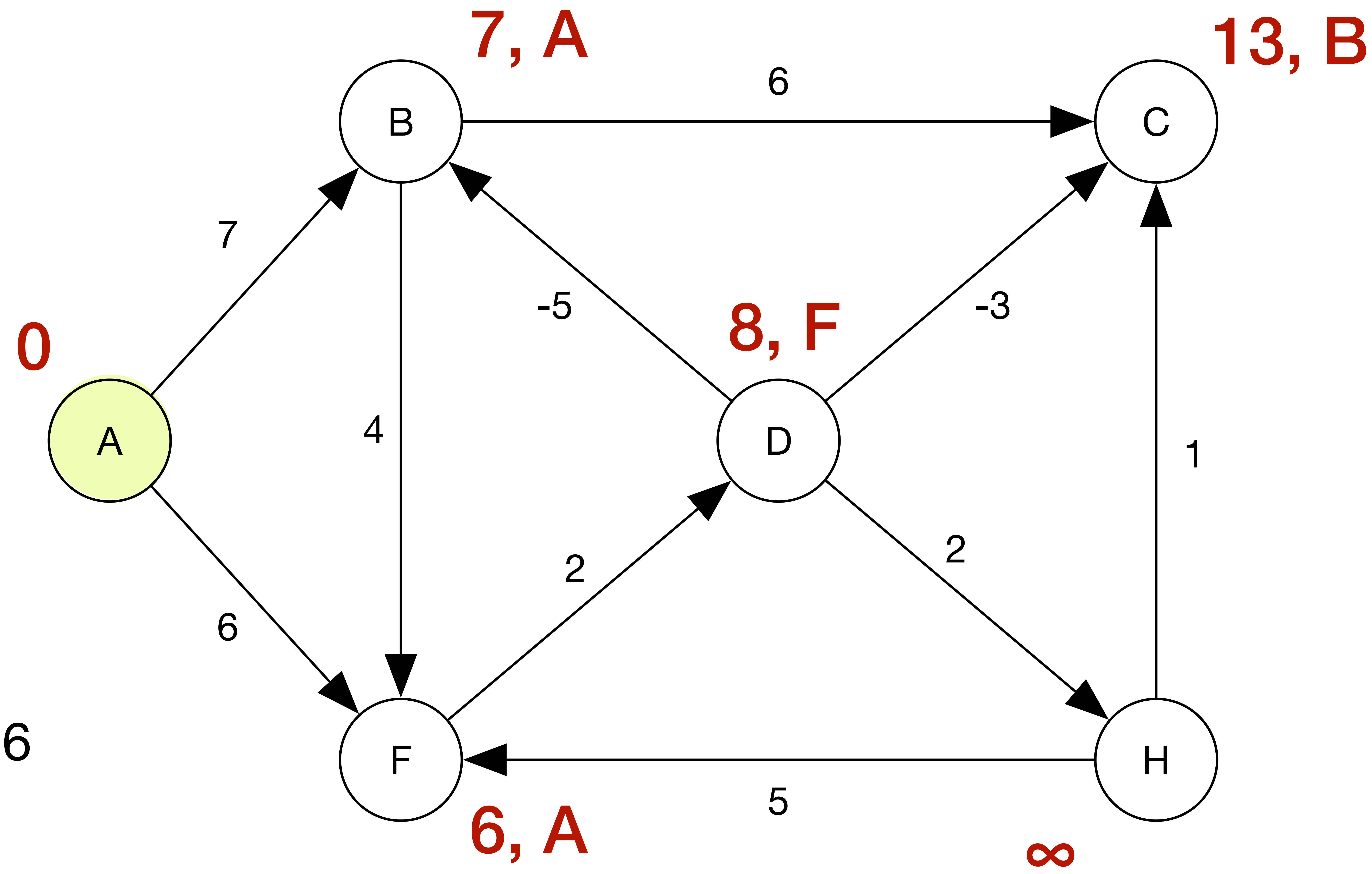

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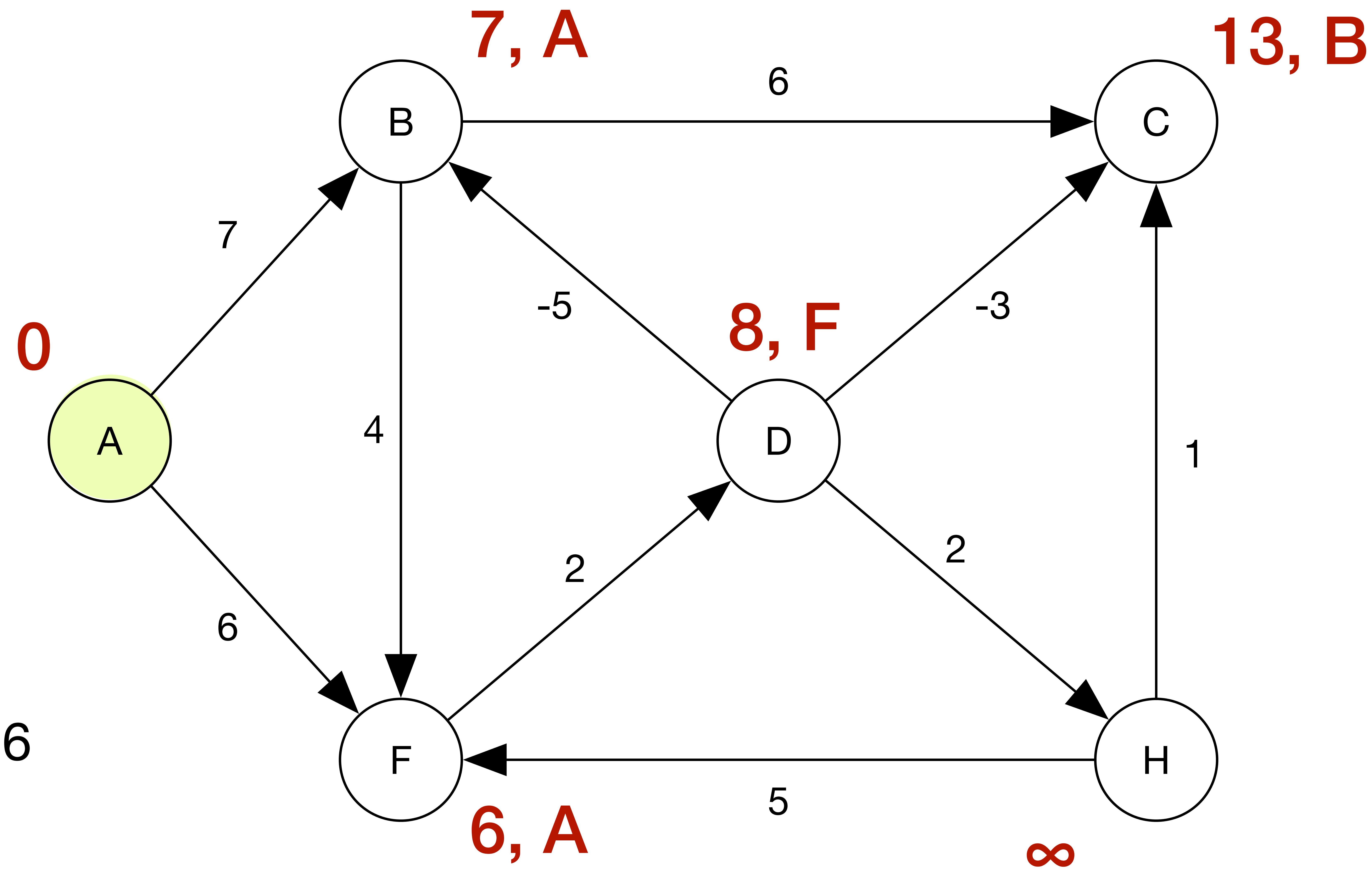

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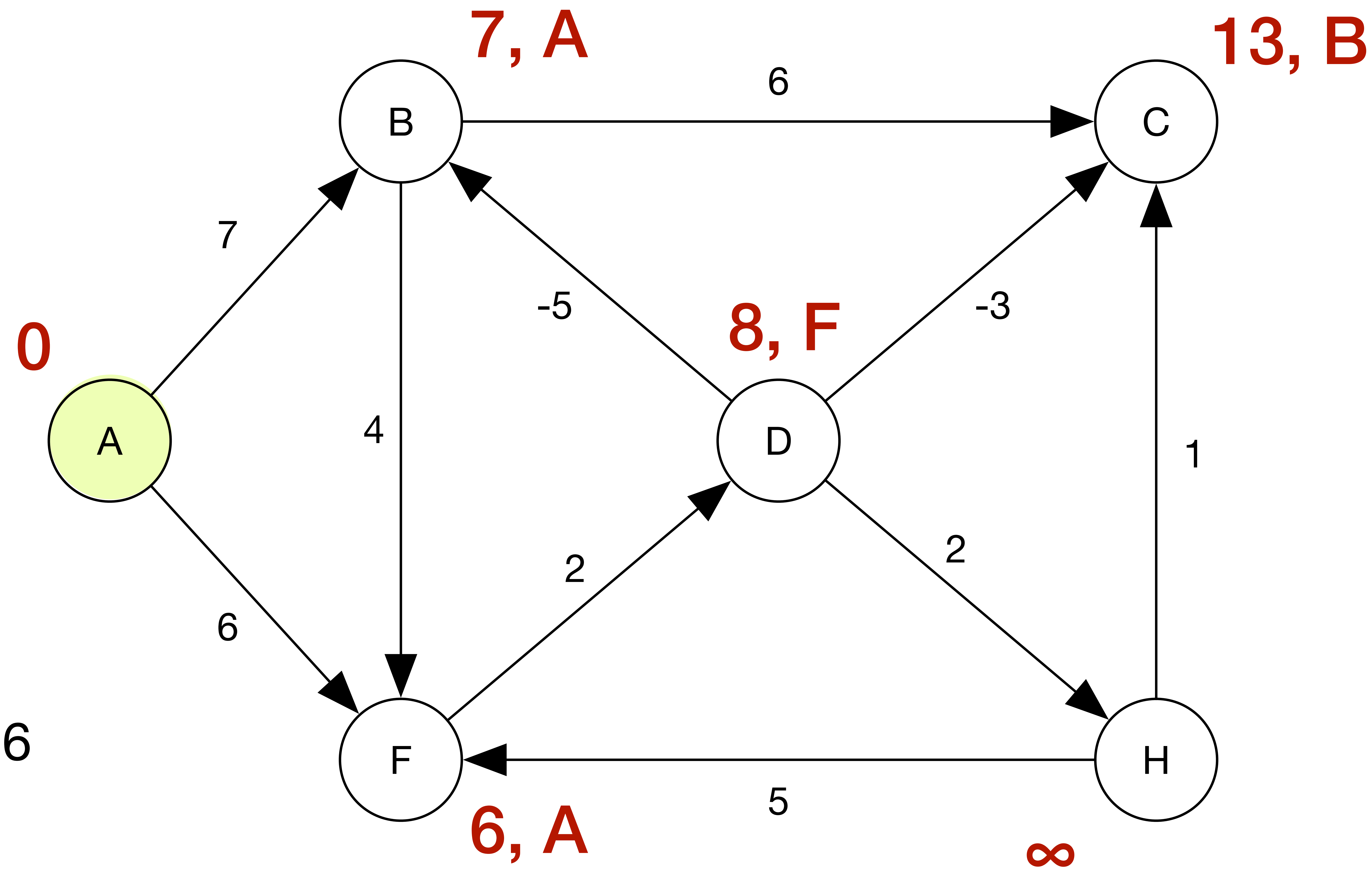

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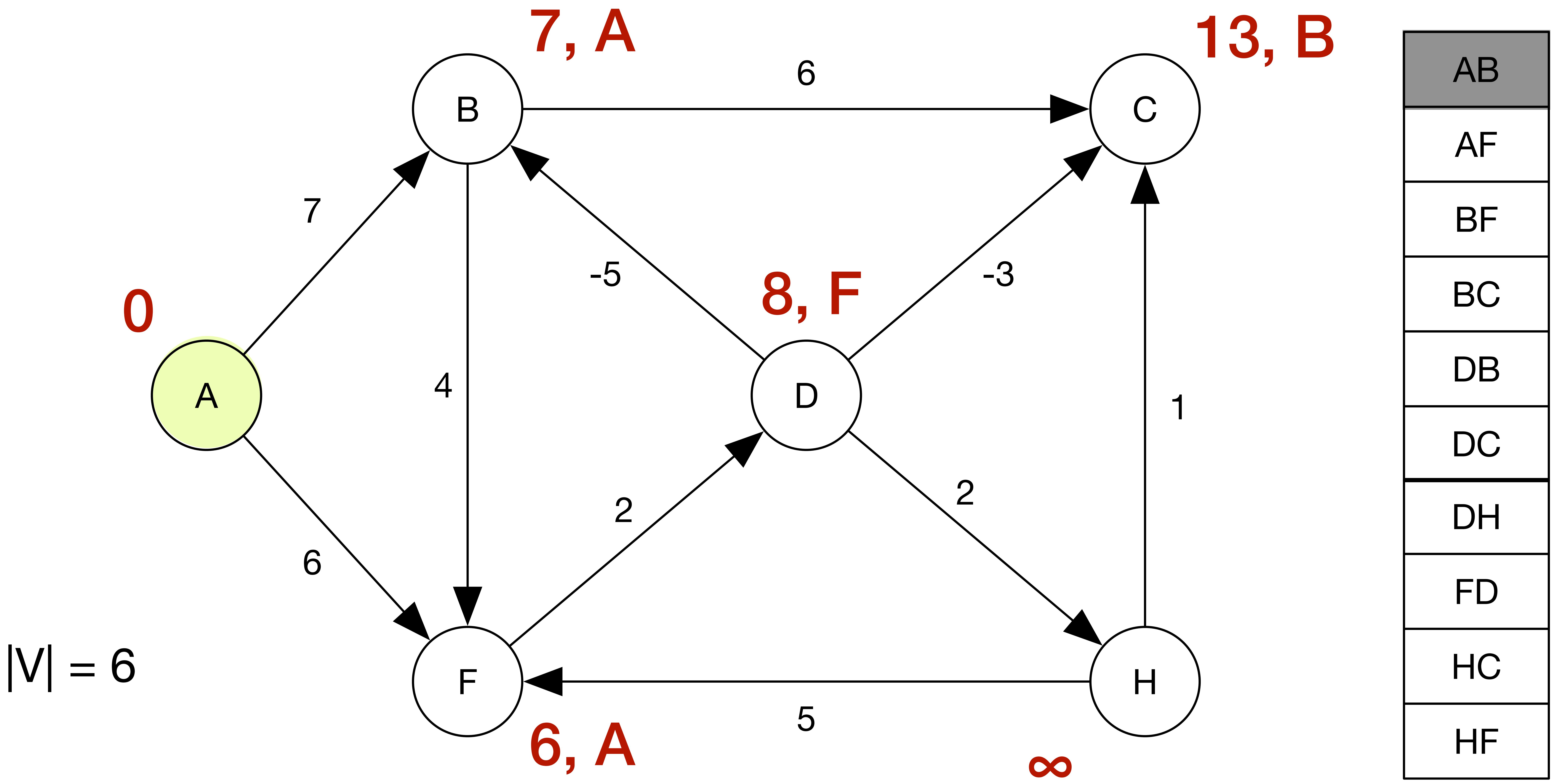

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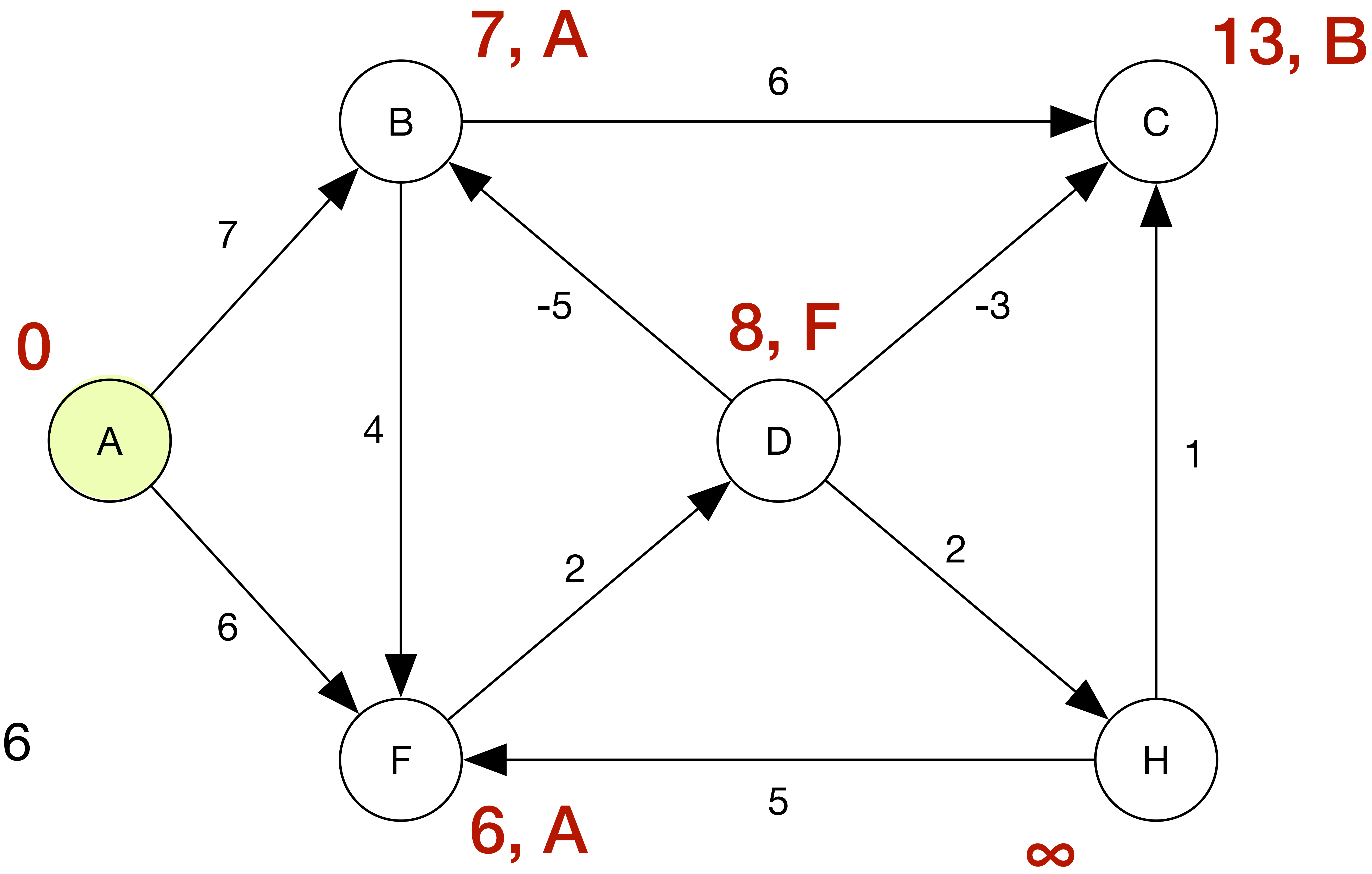

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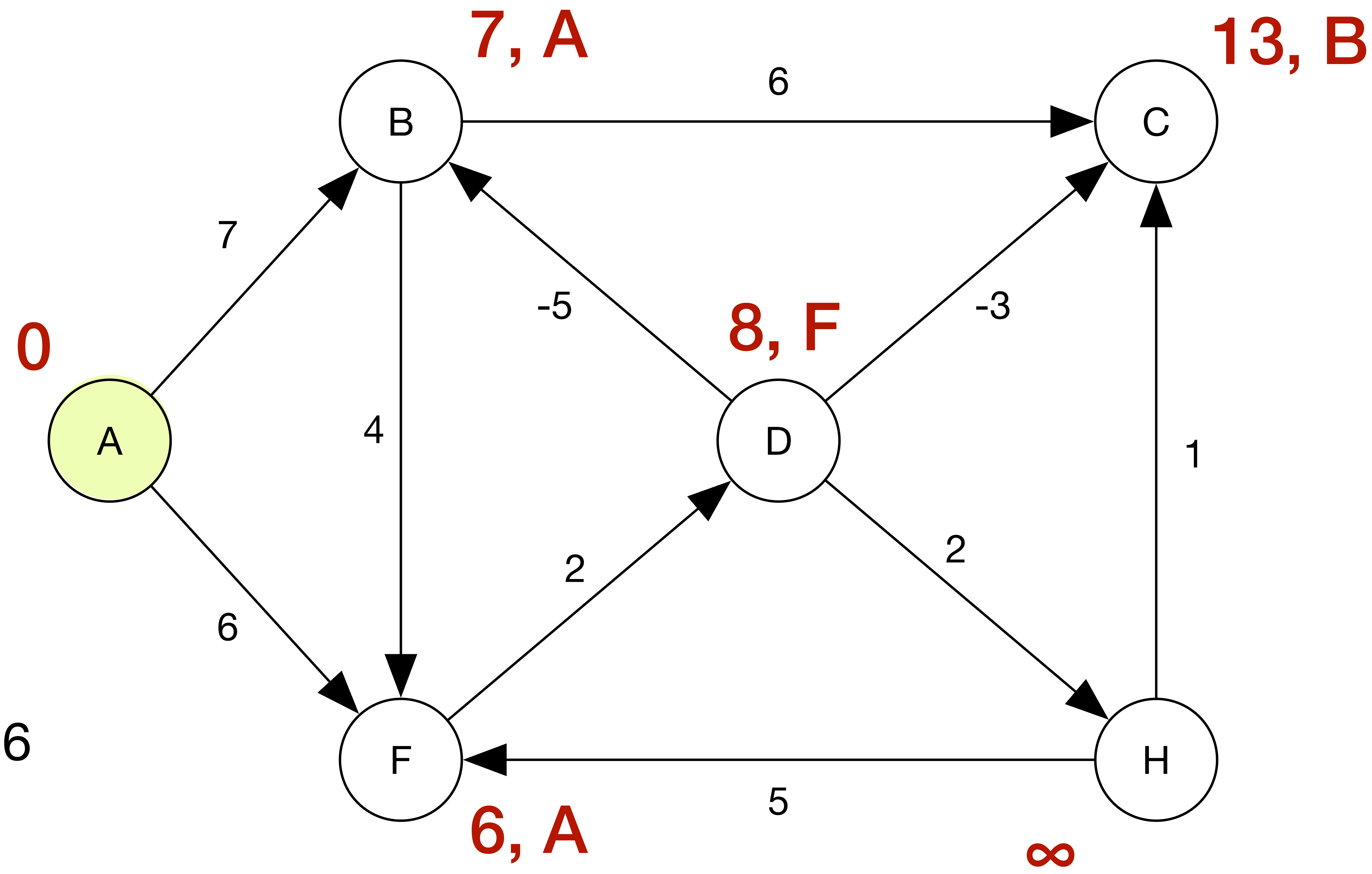

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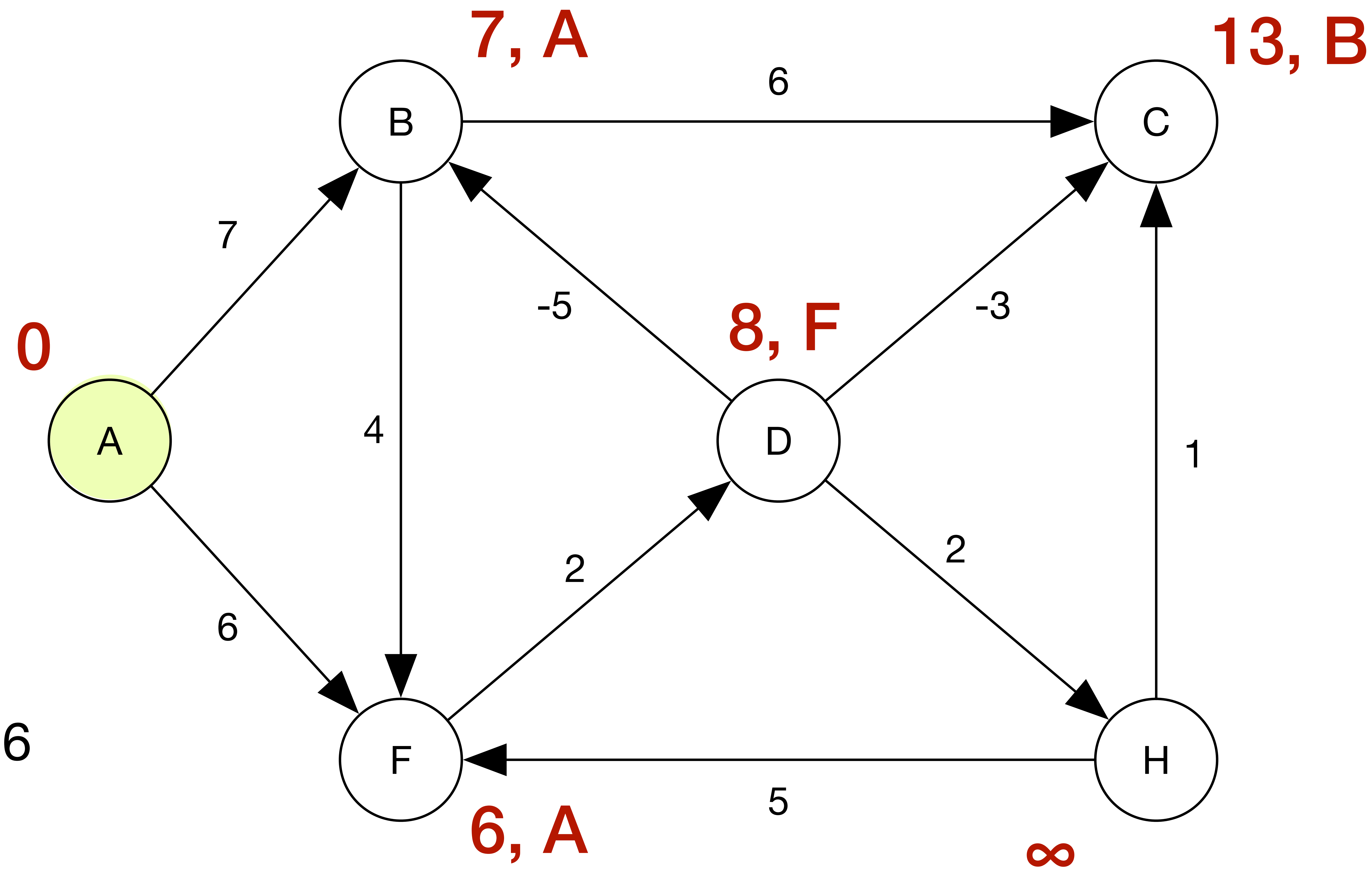

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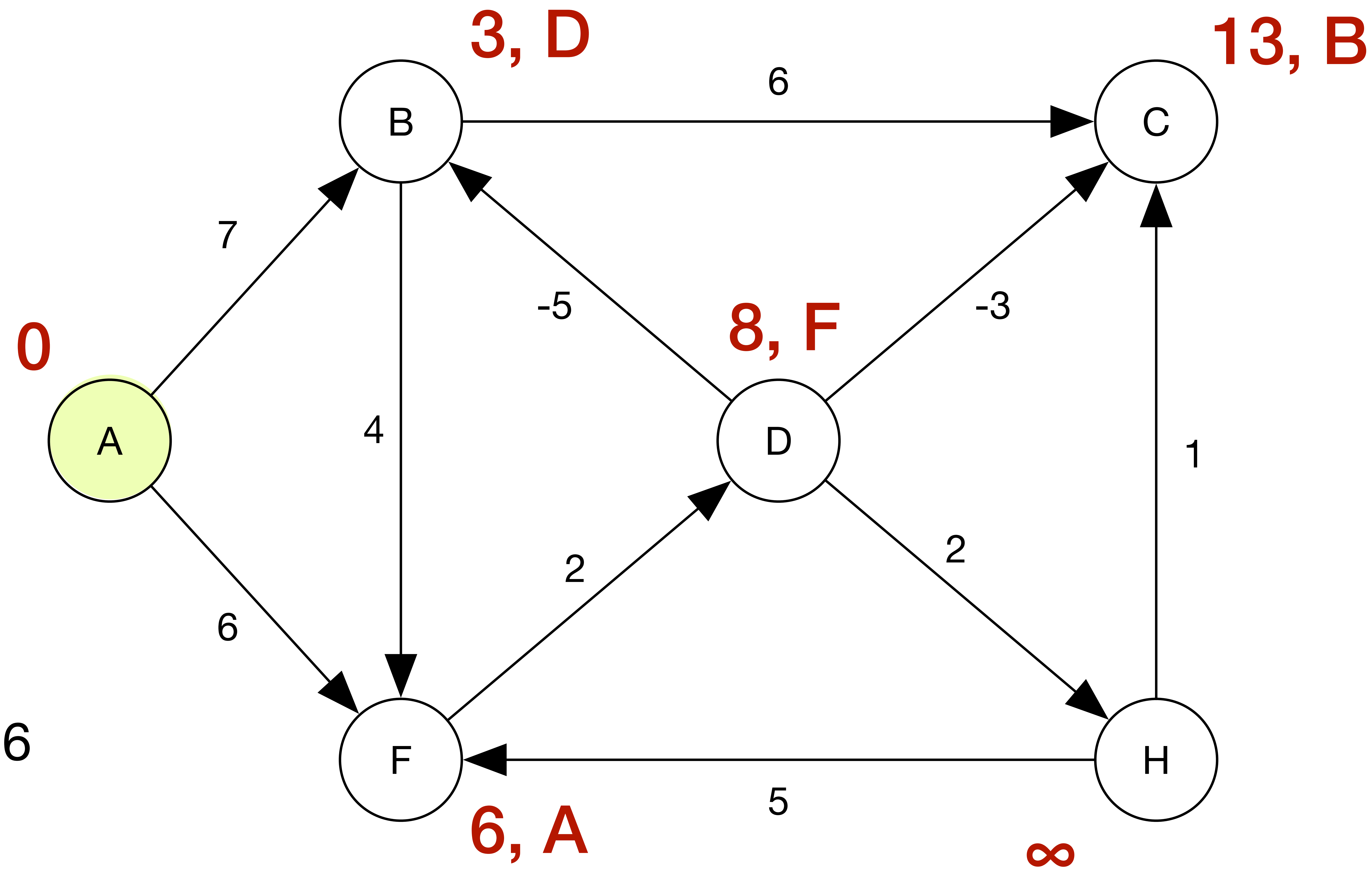

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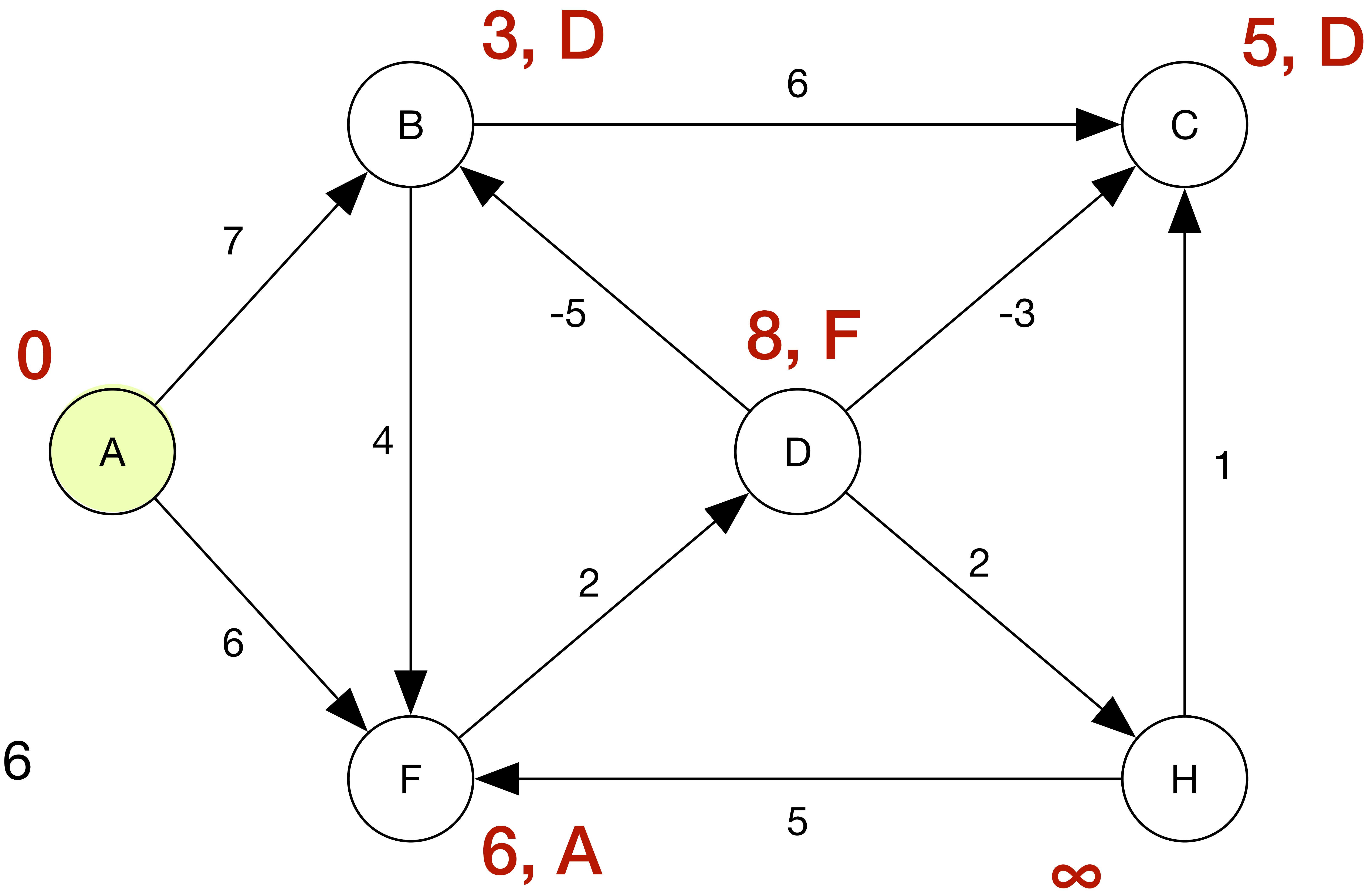

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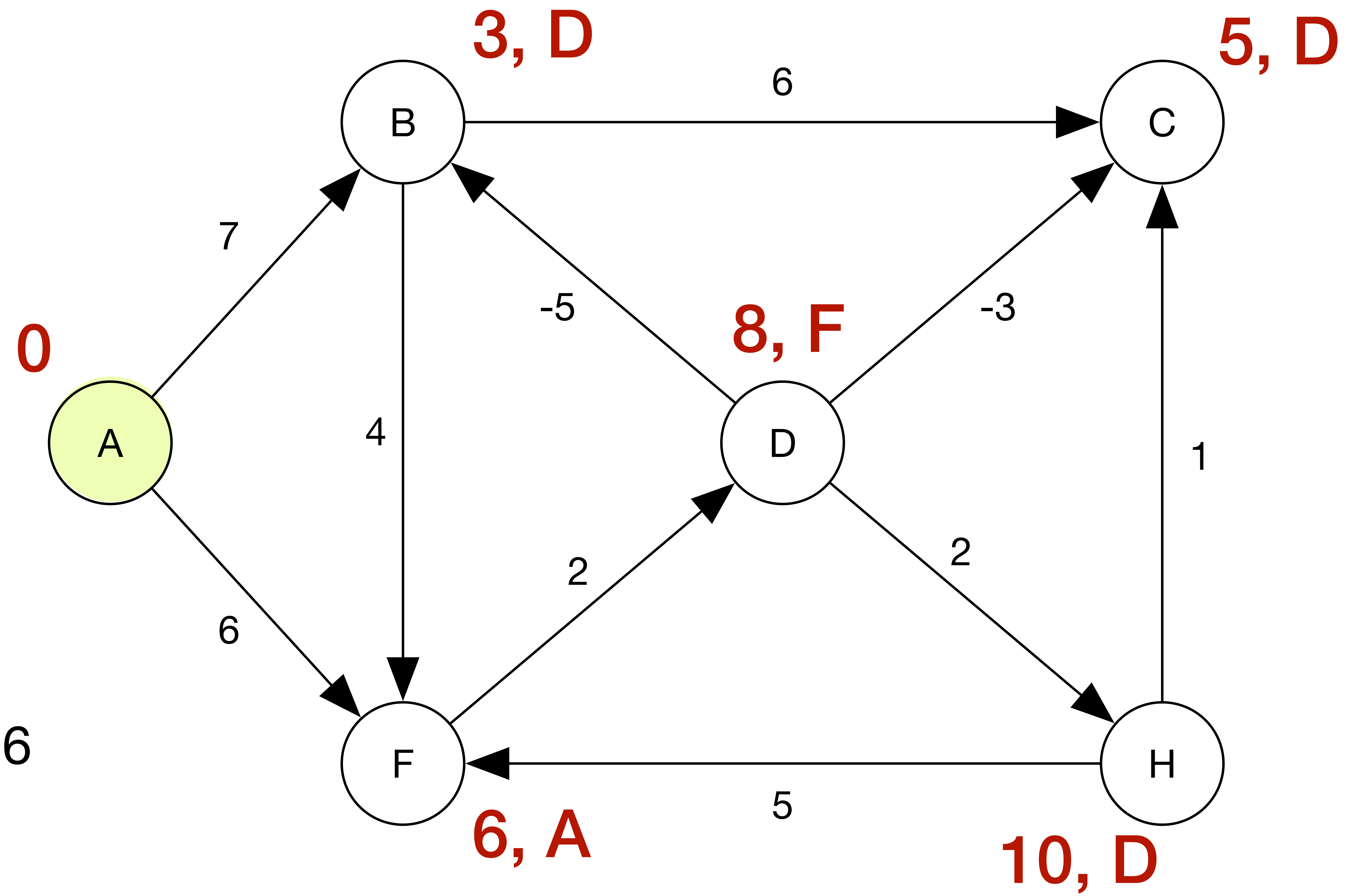

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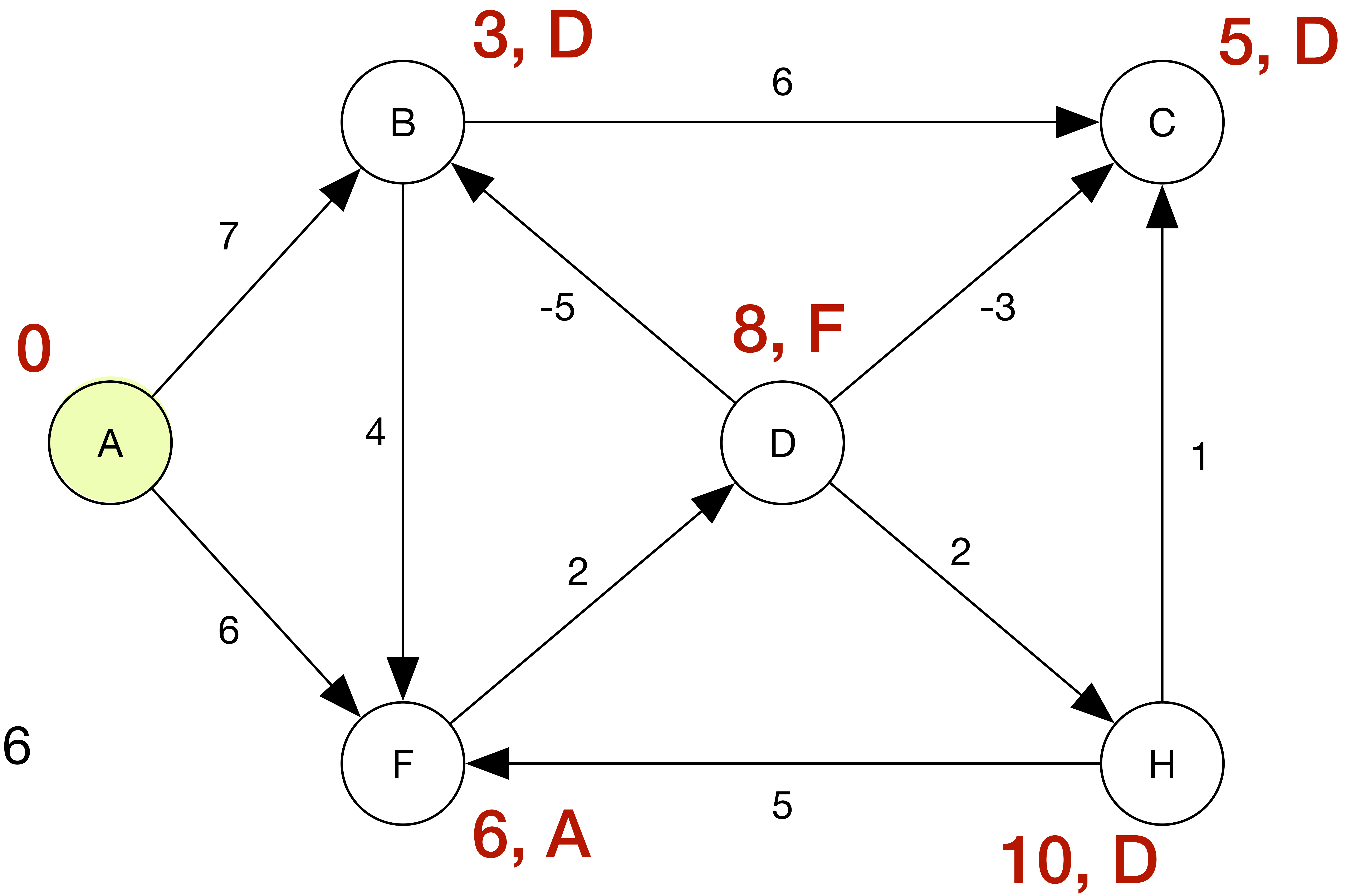

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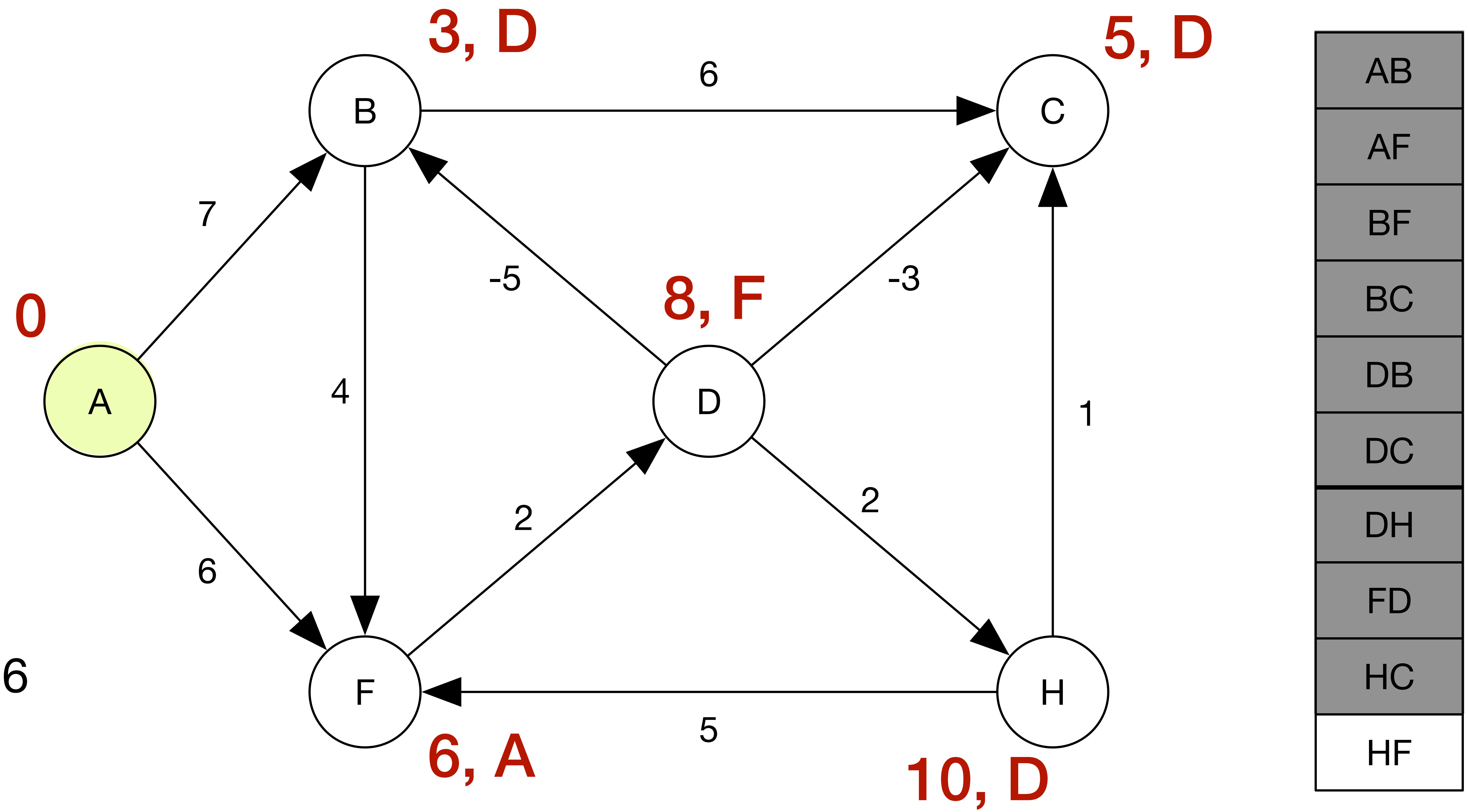

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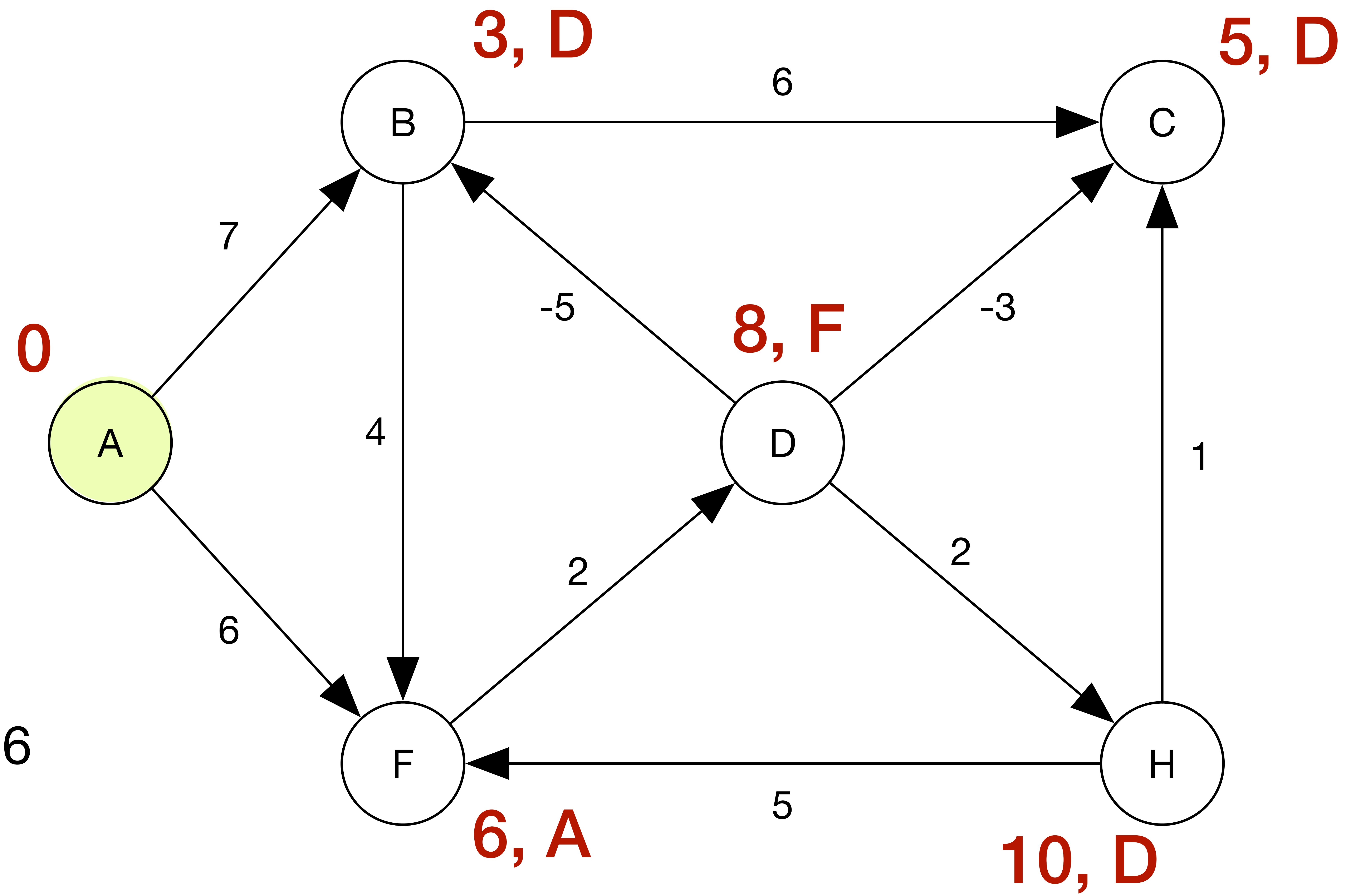

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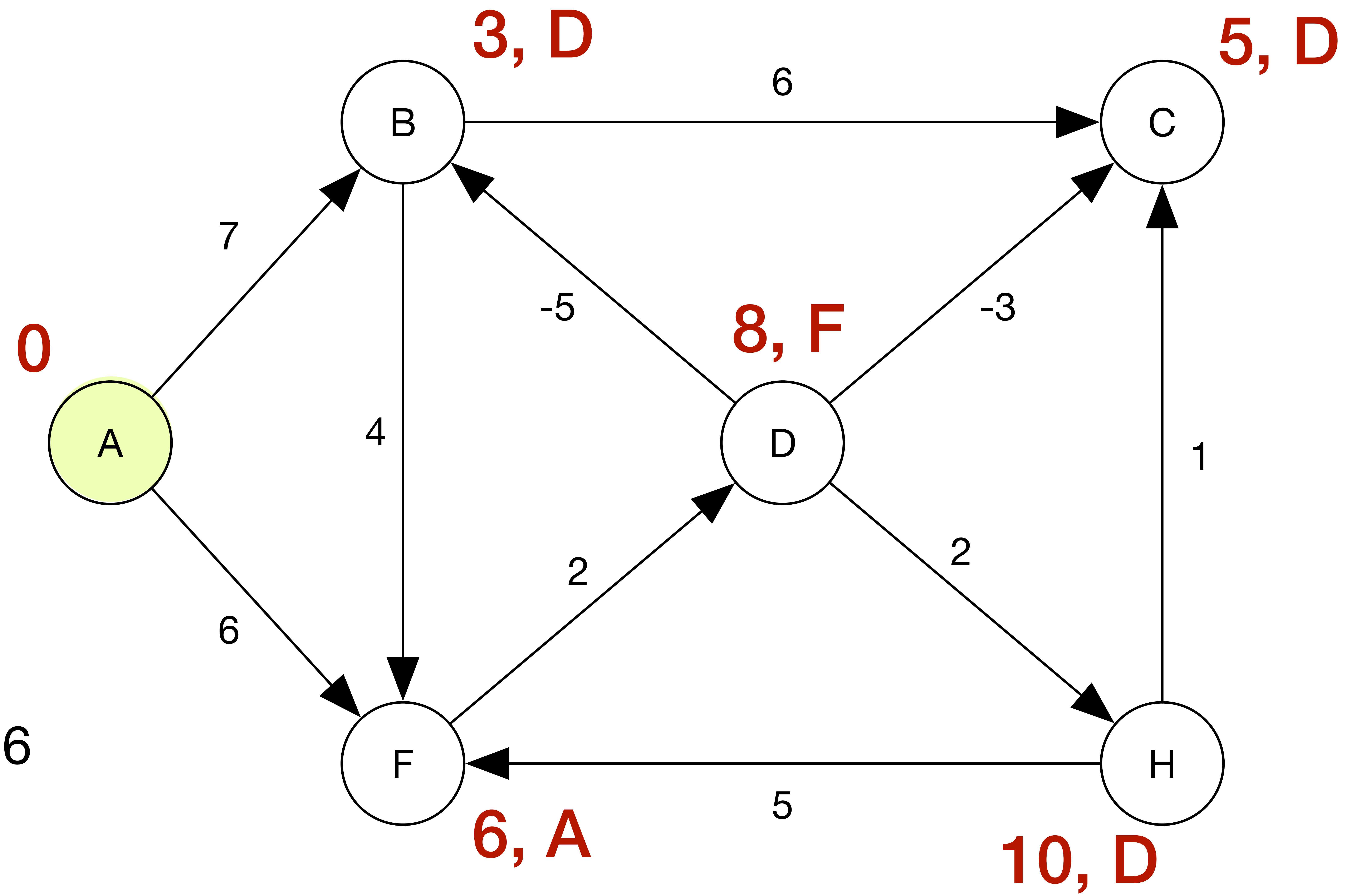

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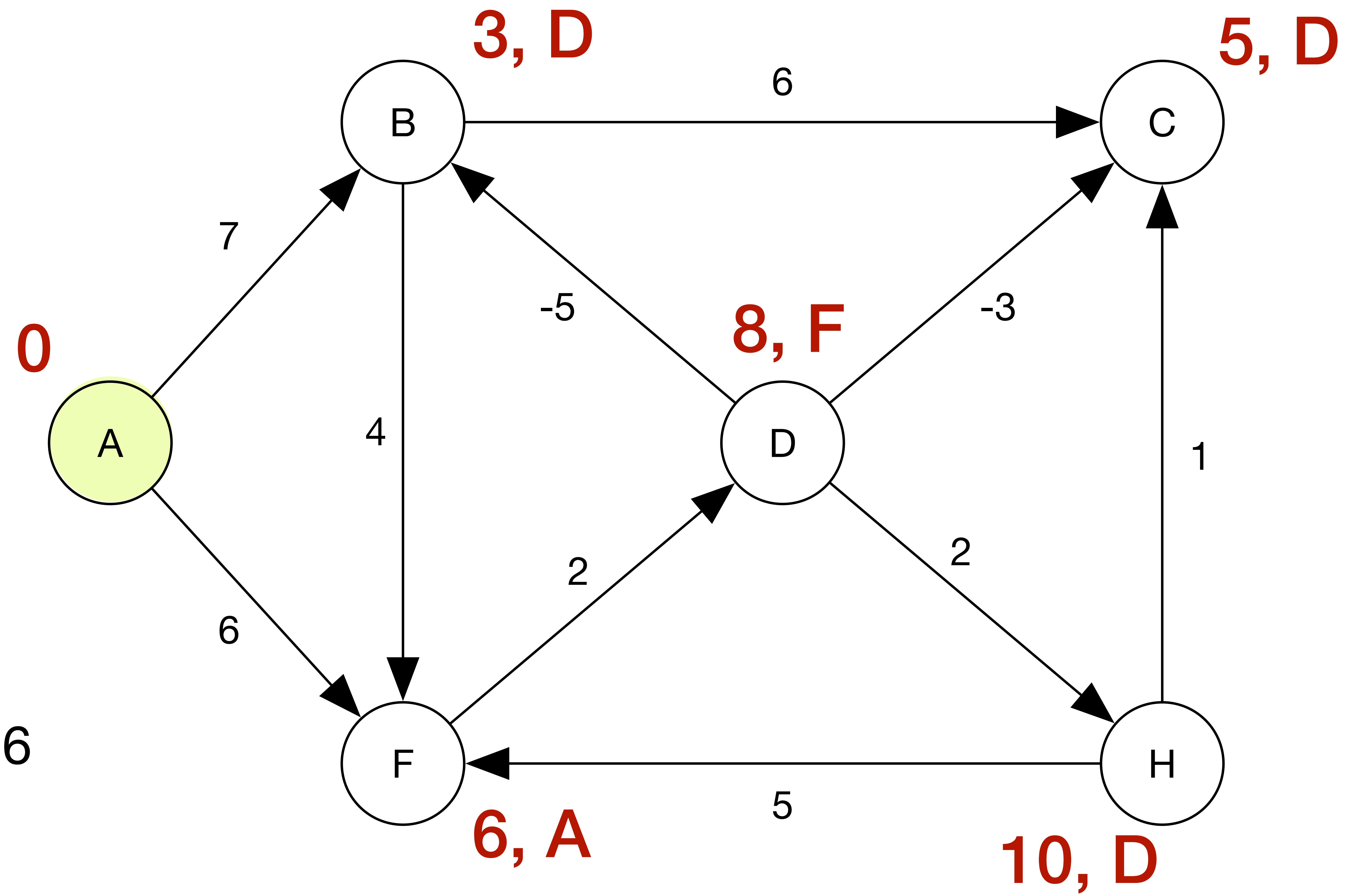

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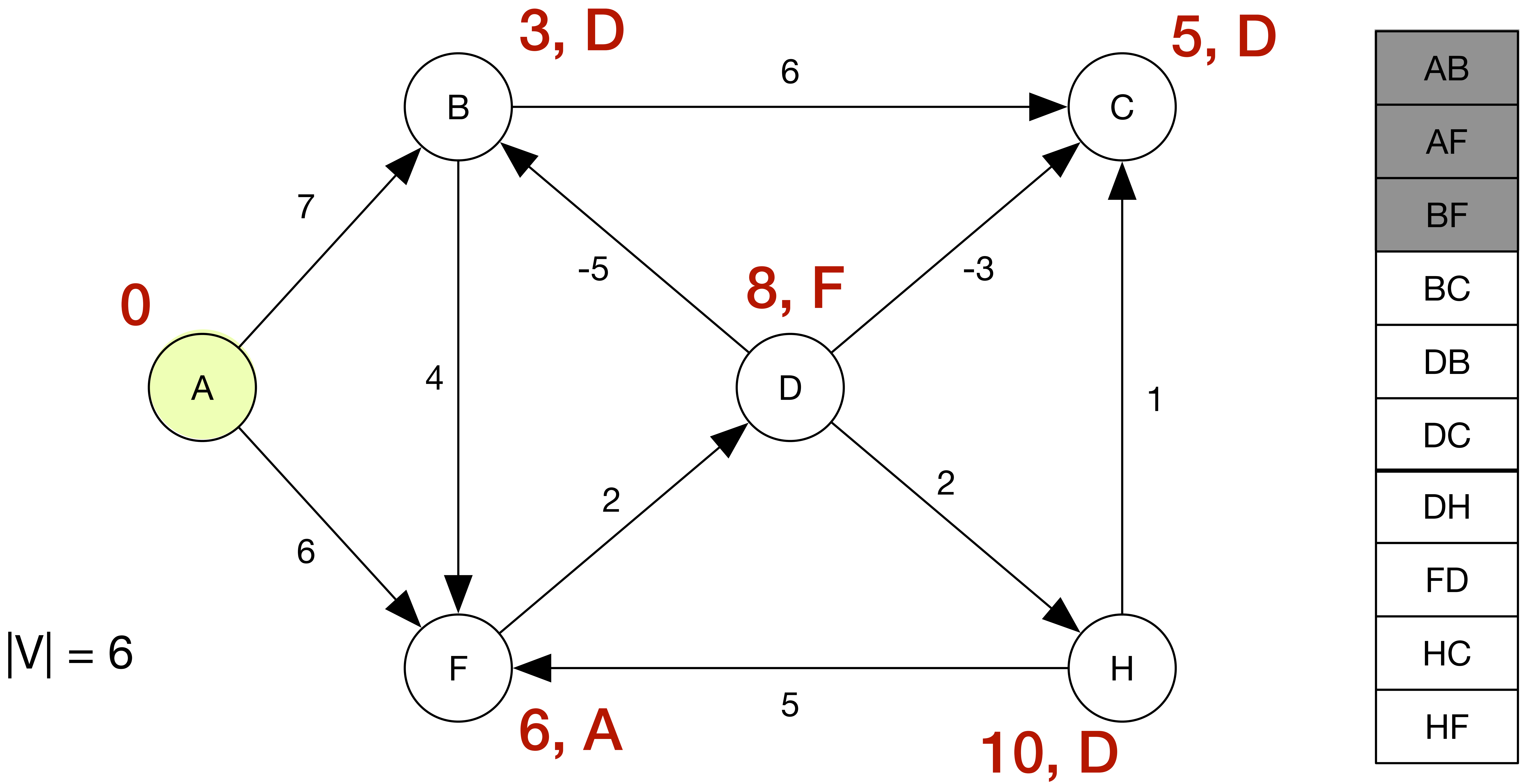

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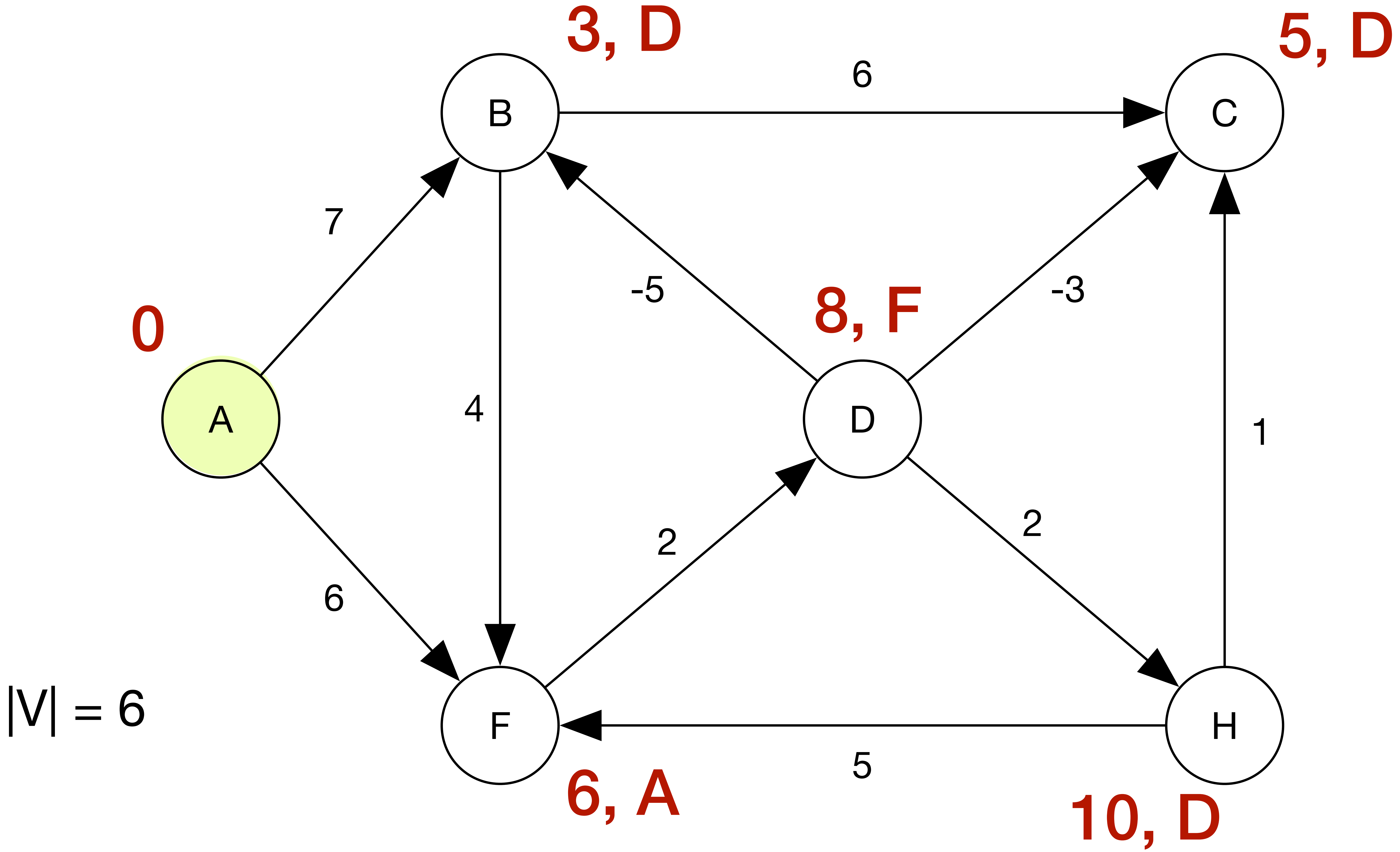

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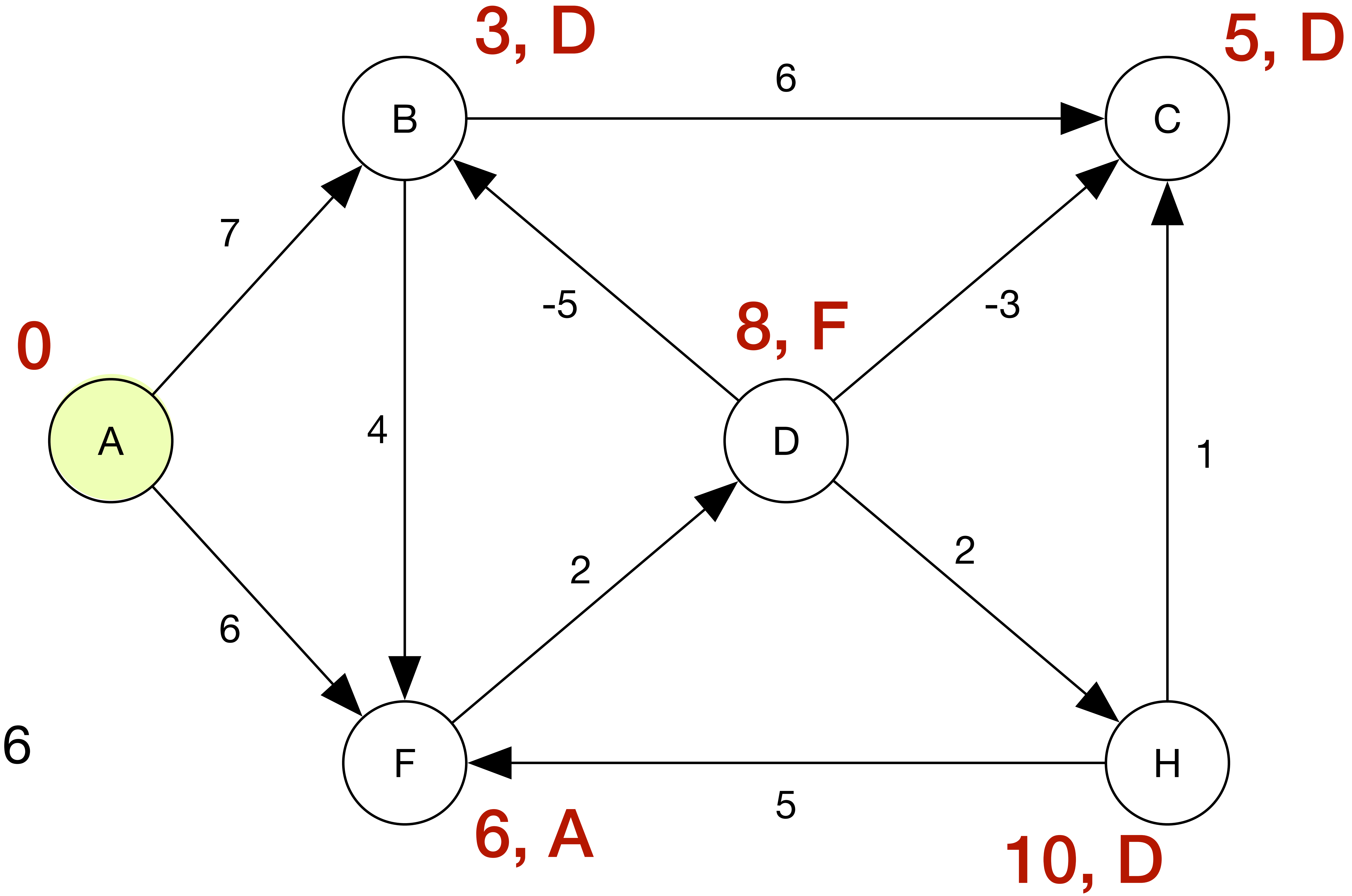

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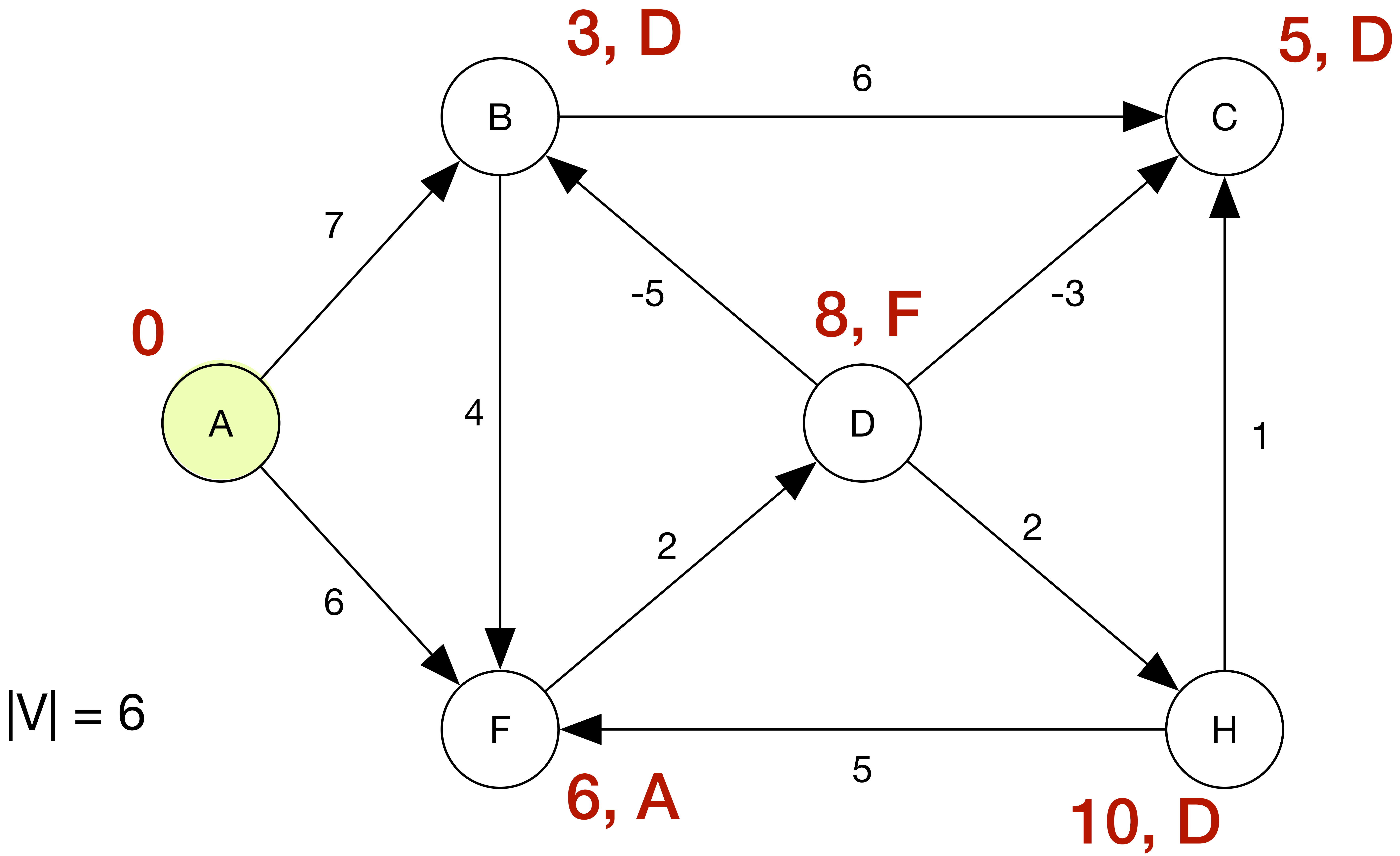

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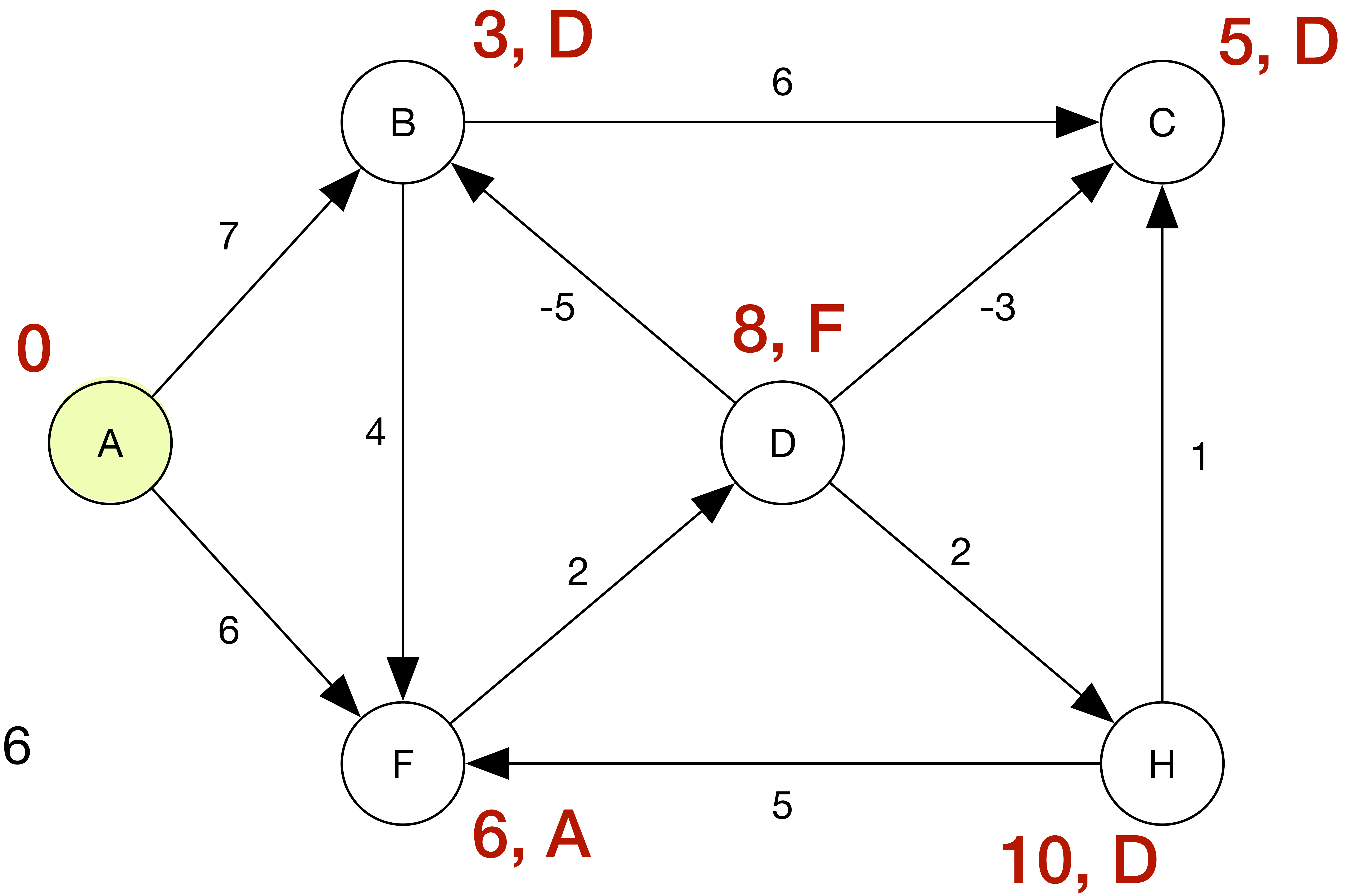

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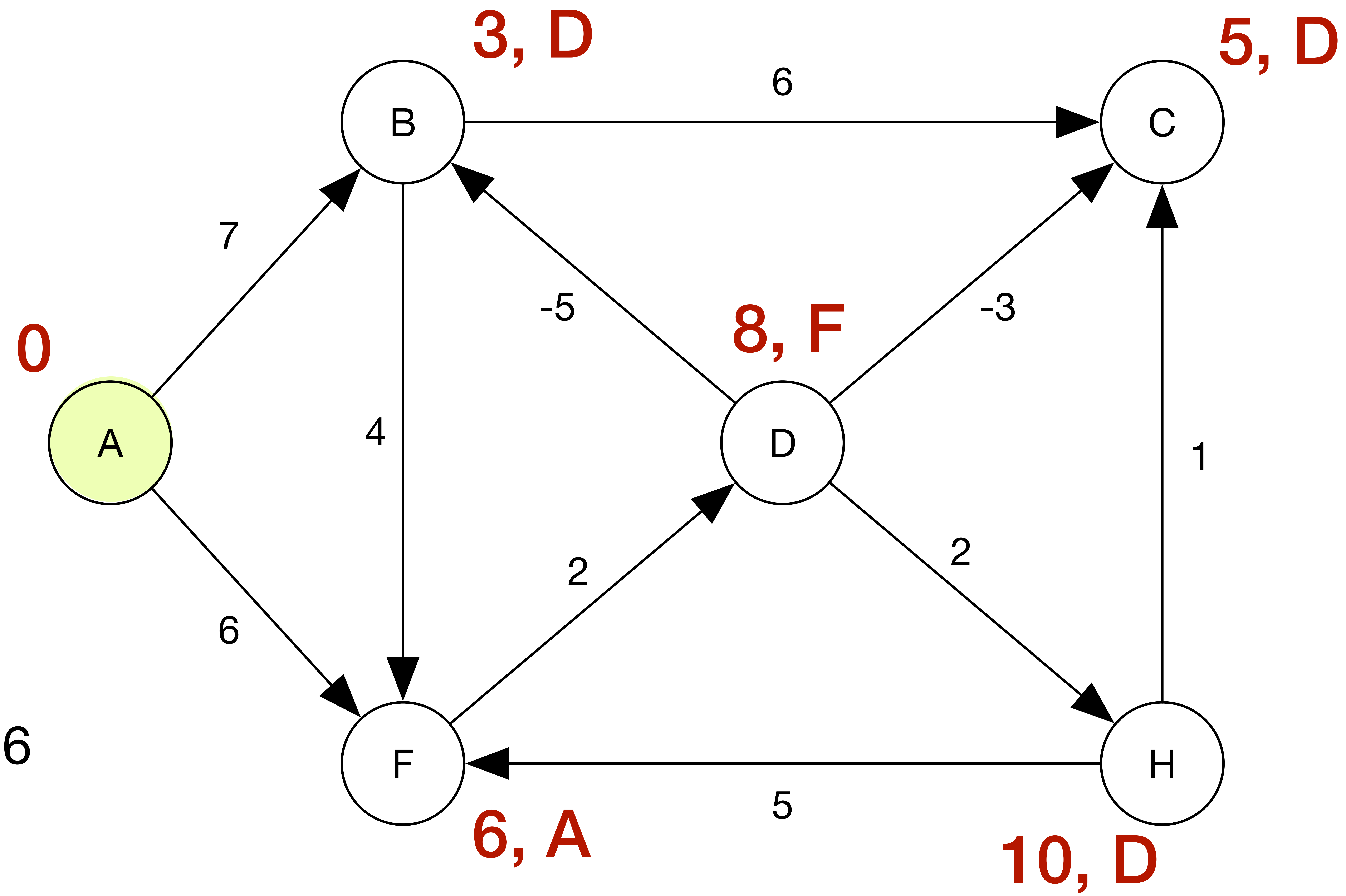

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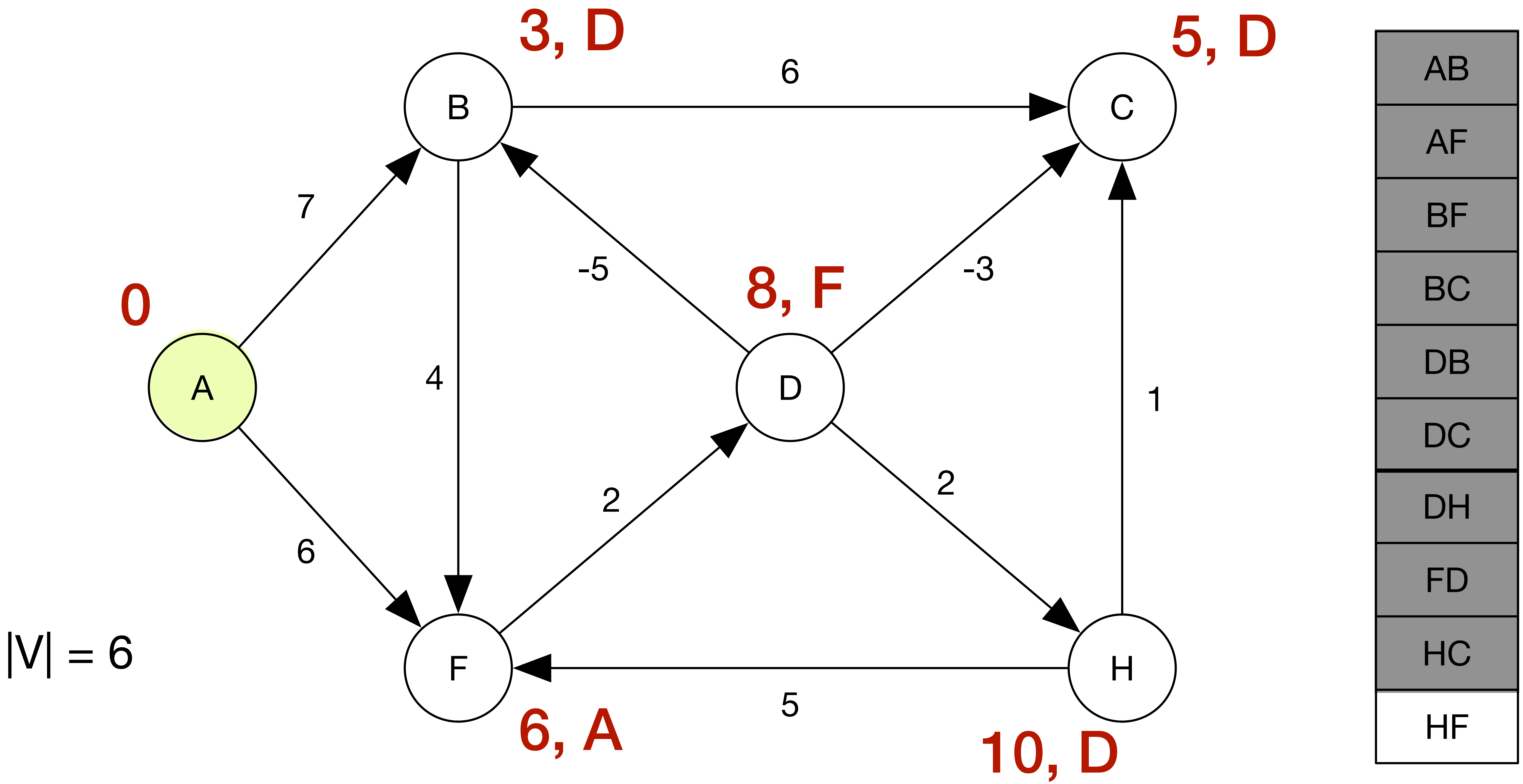

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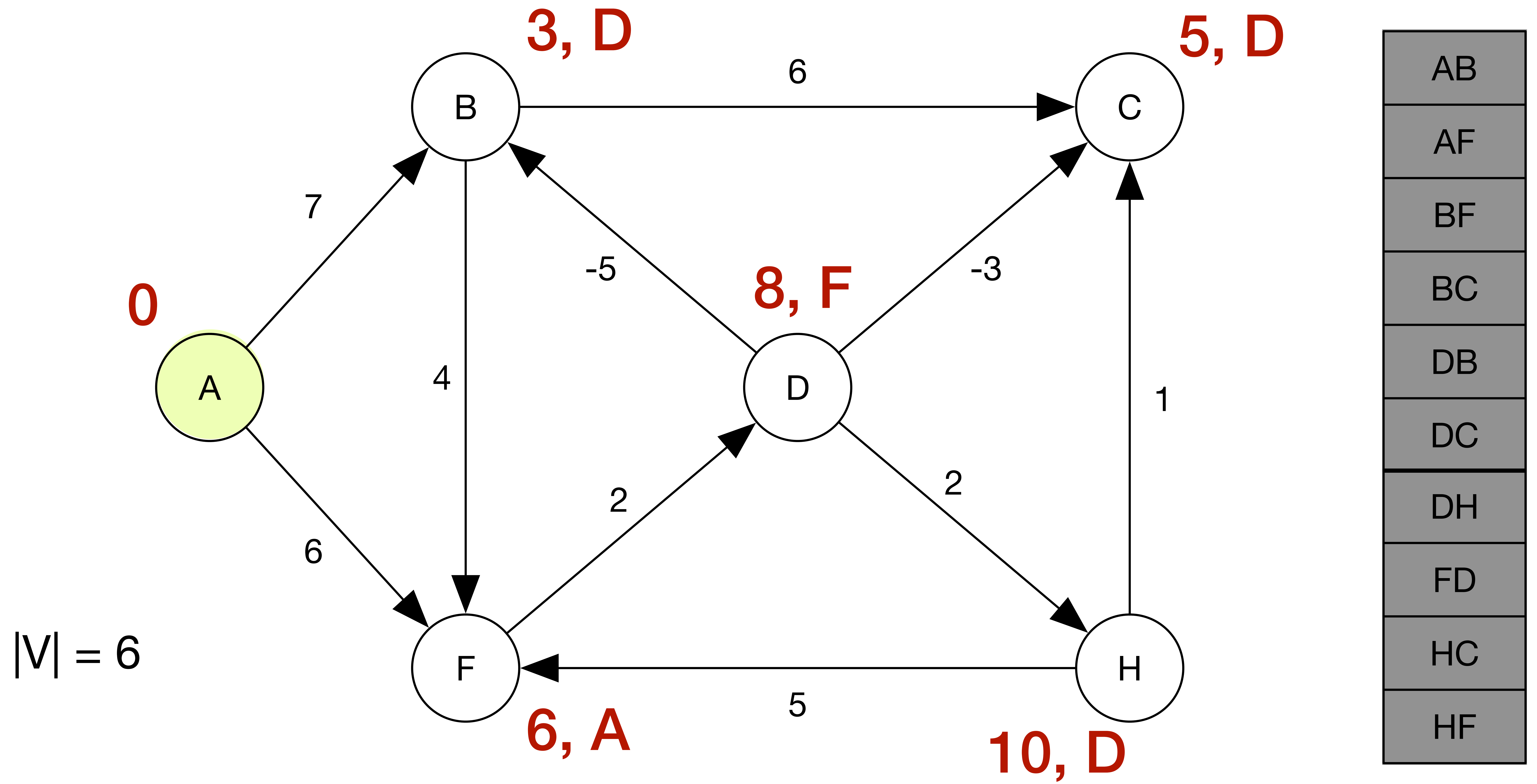
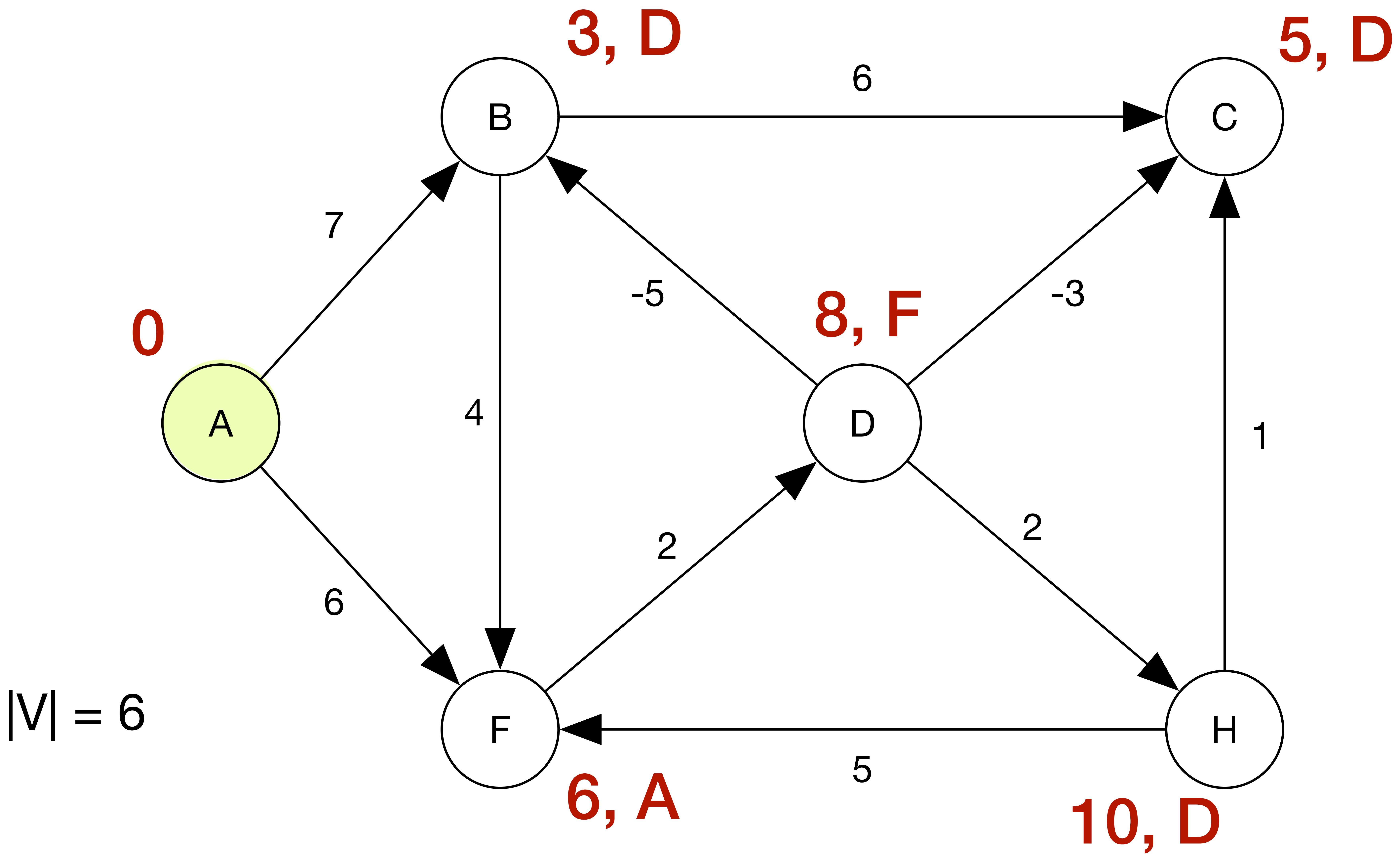


Bellman-Ford

Check for negative cycles



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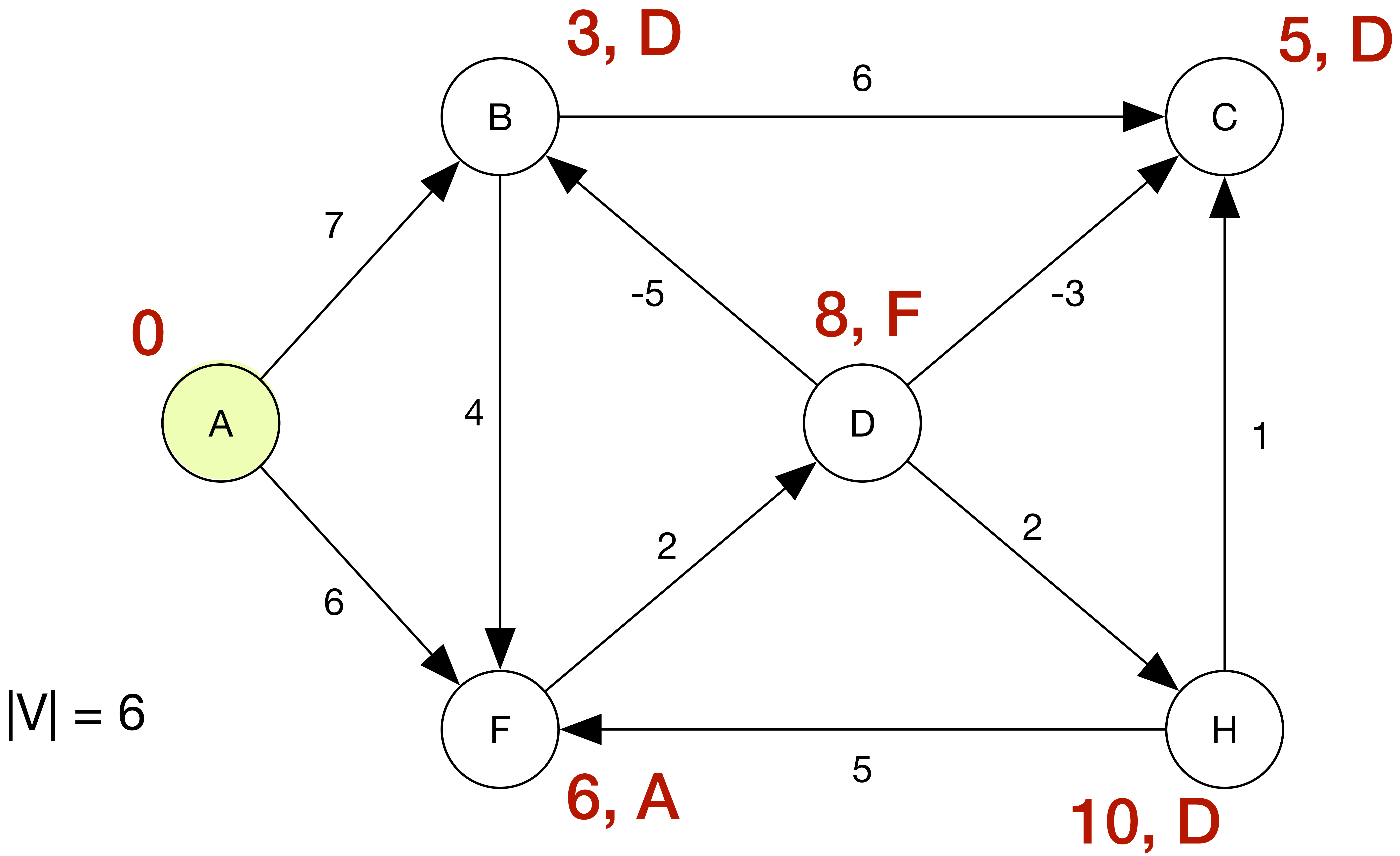

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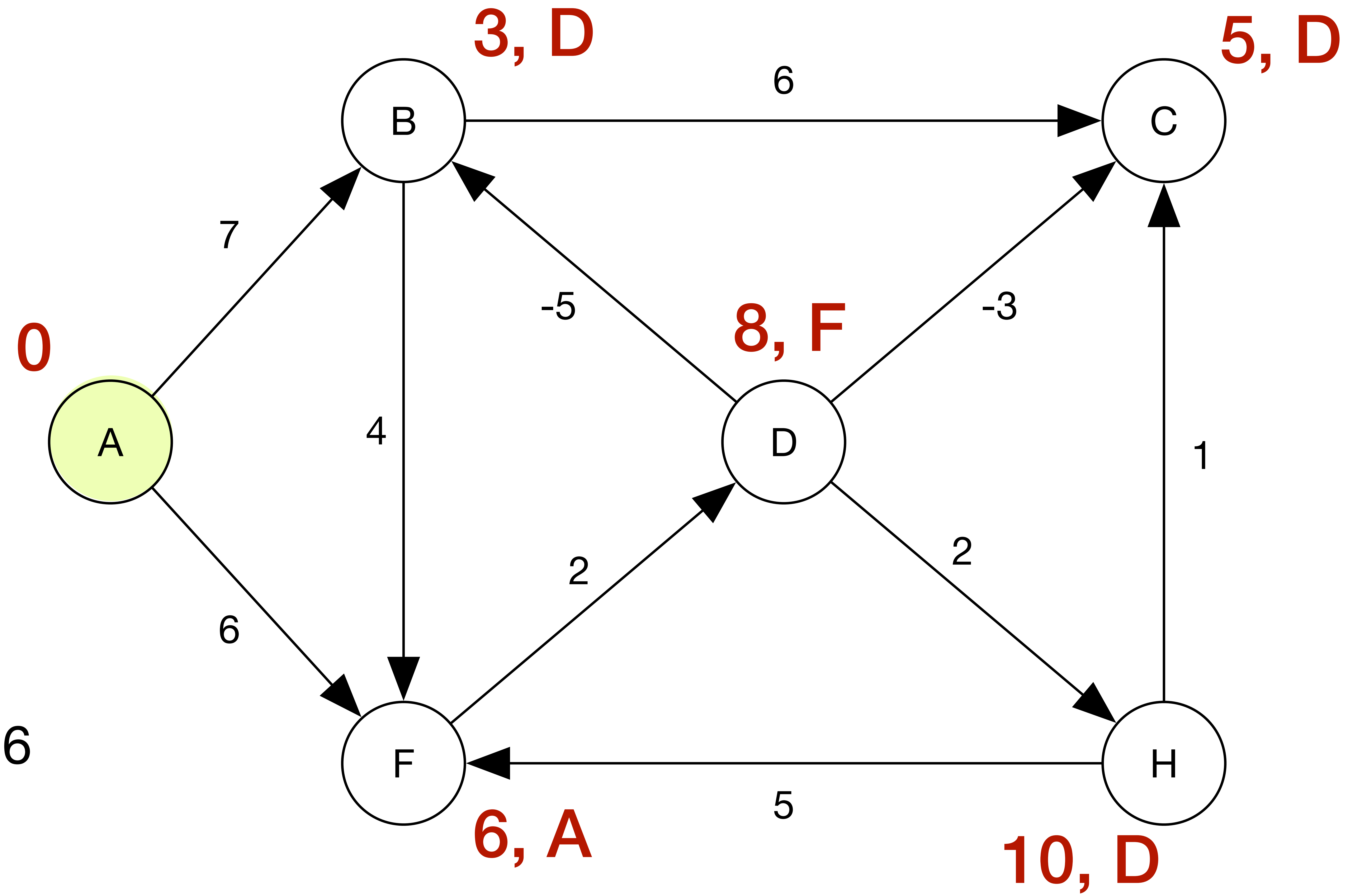

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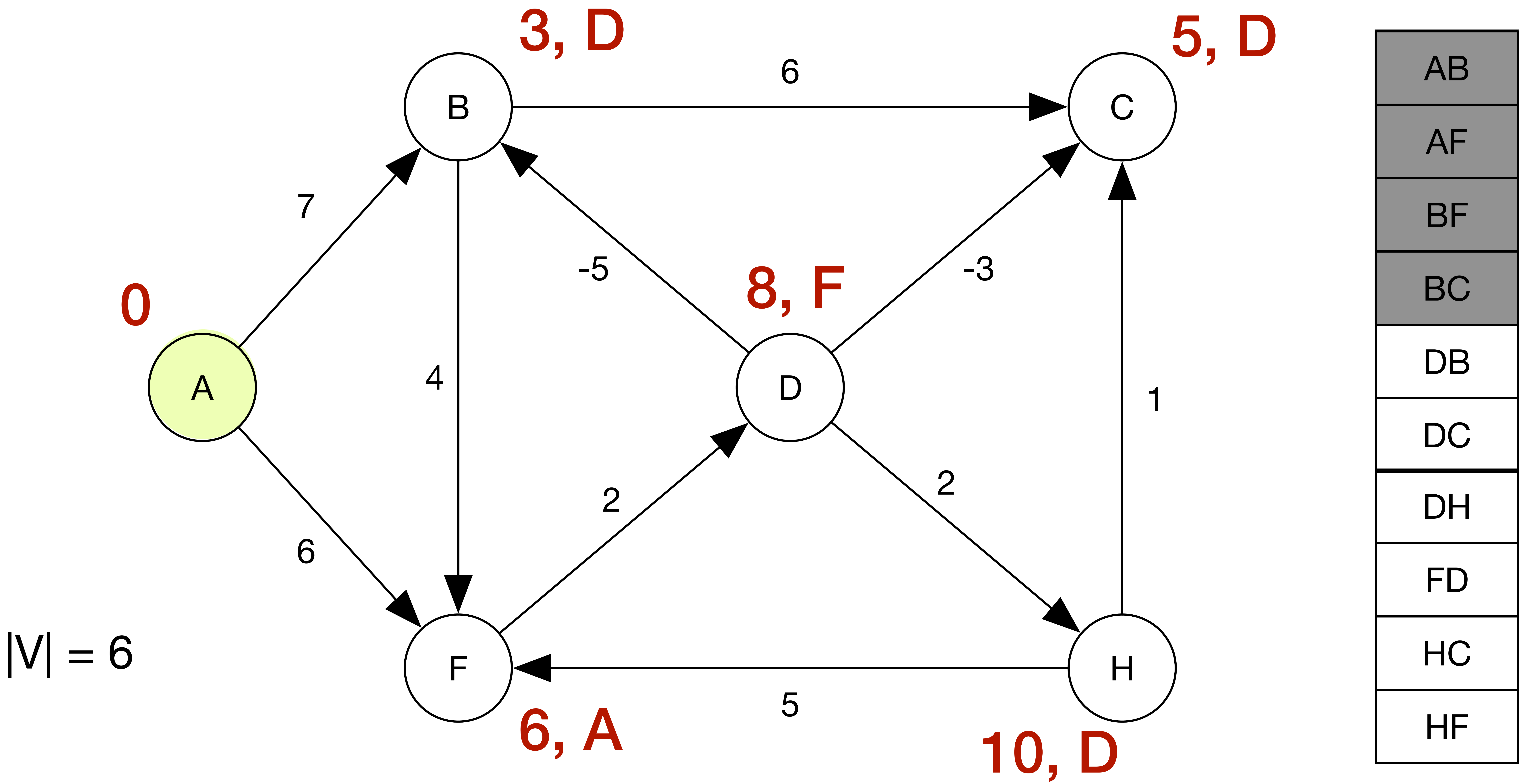

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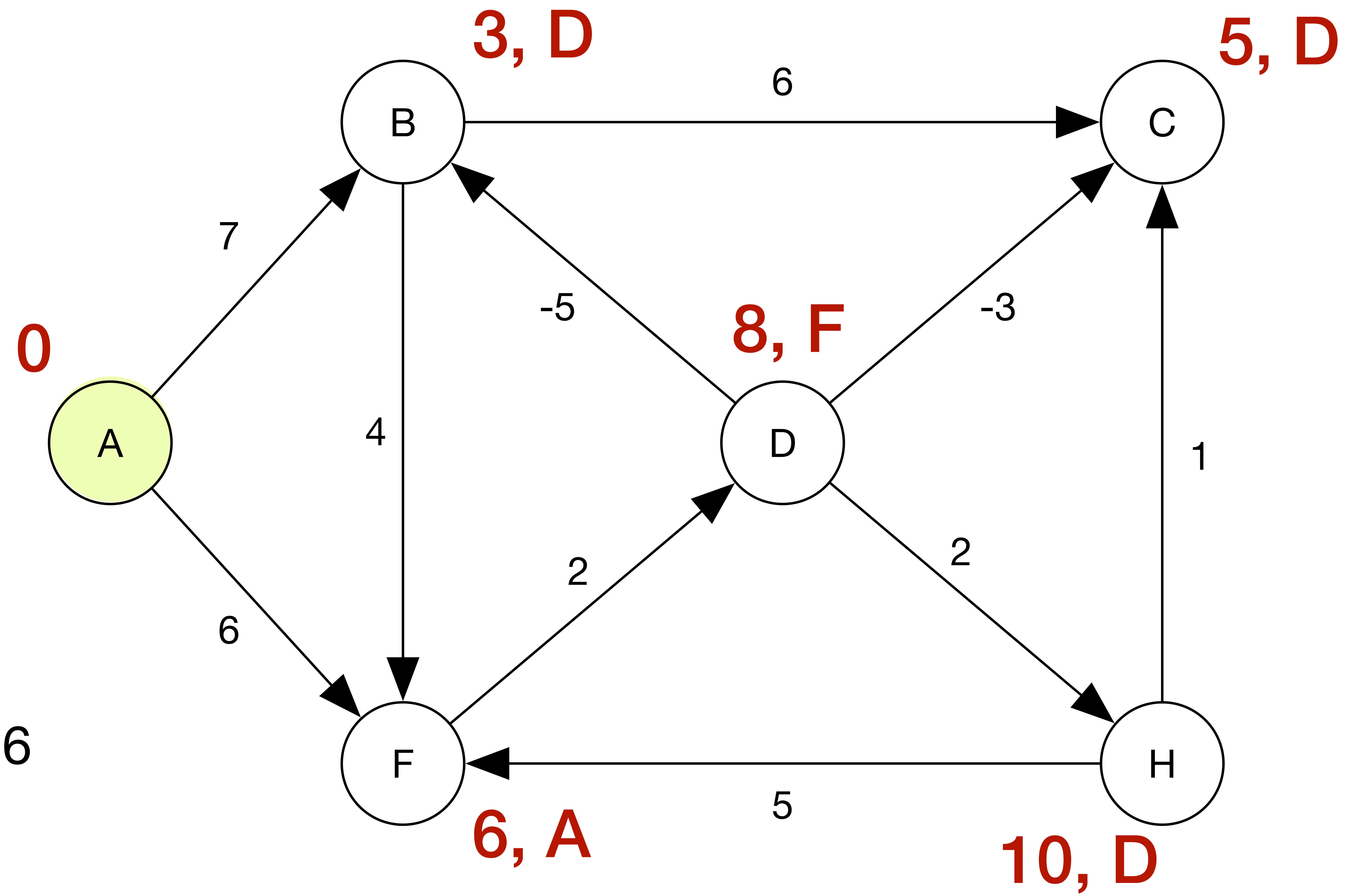

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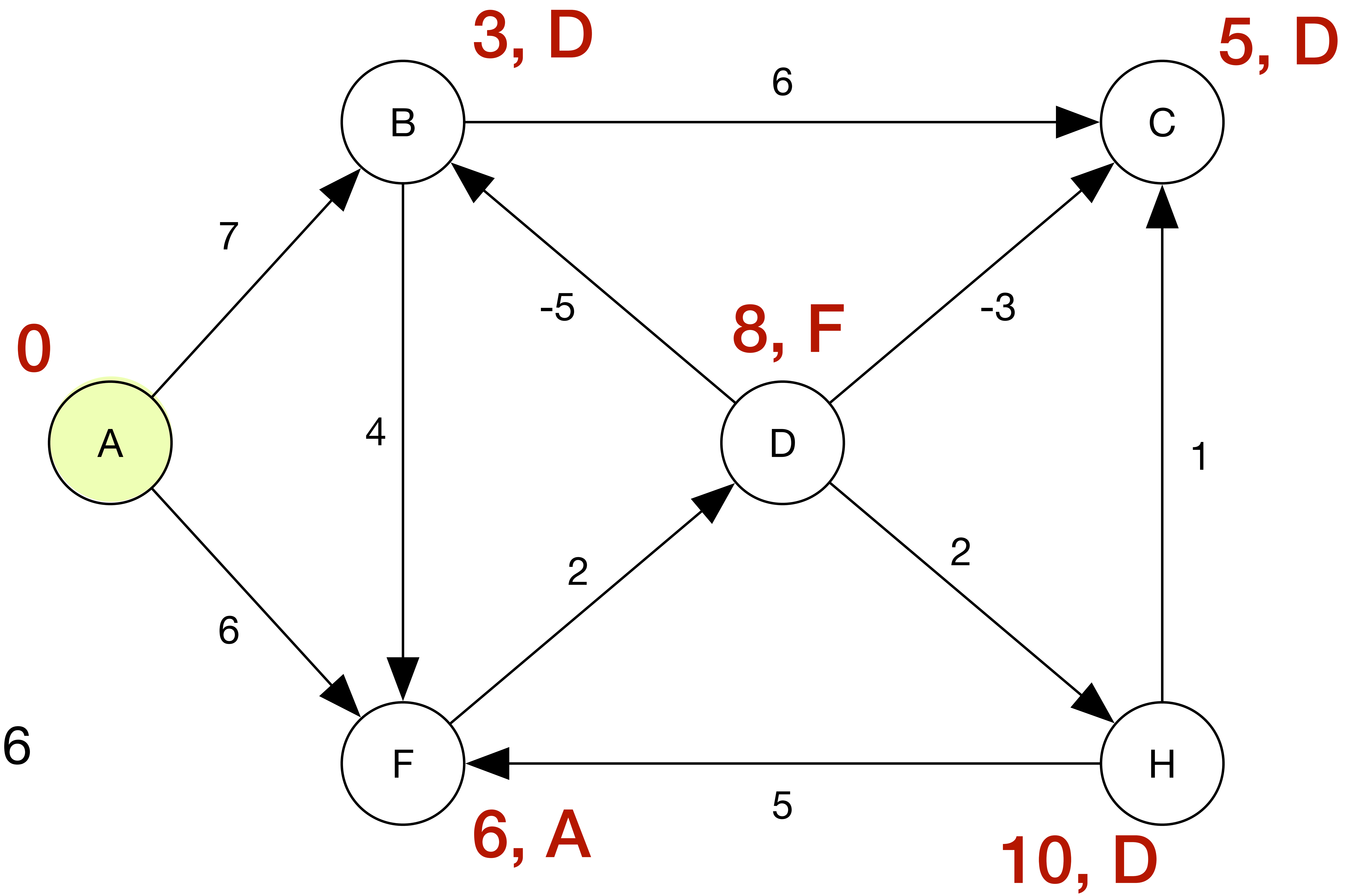

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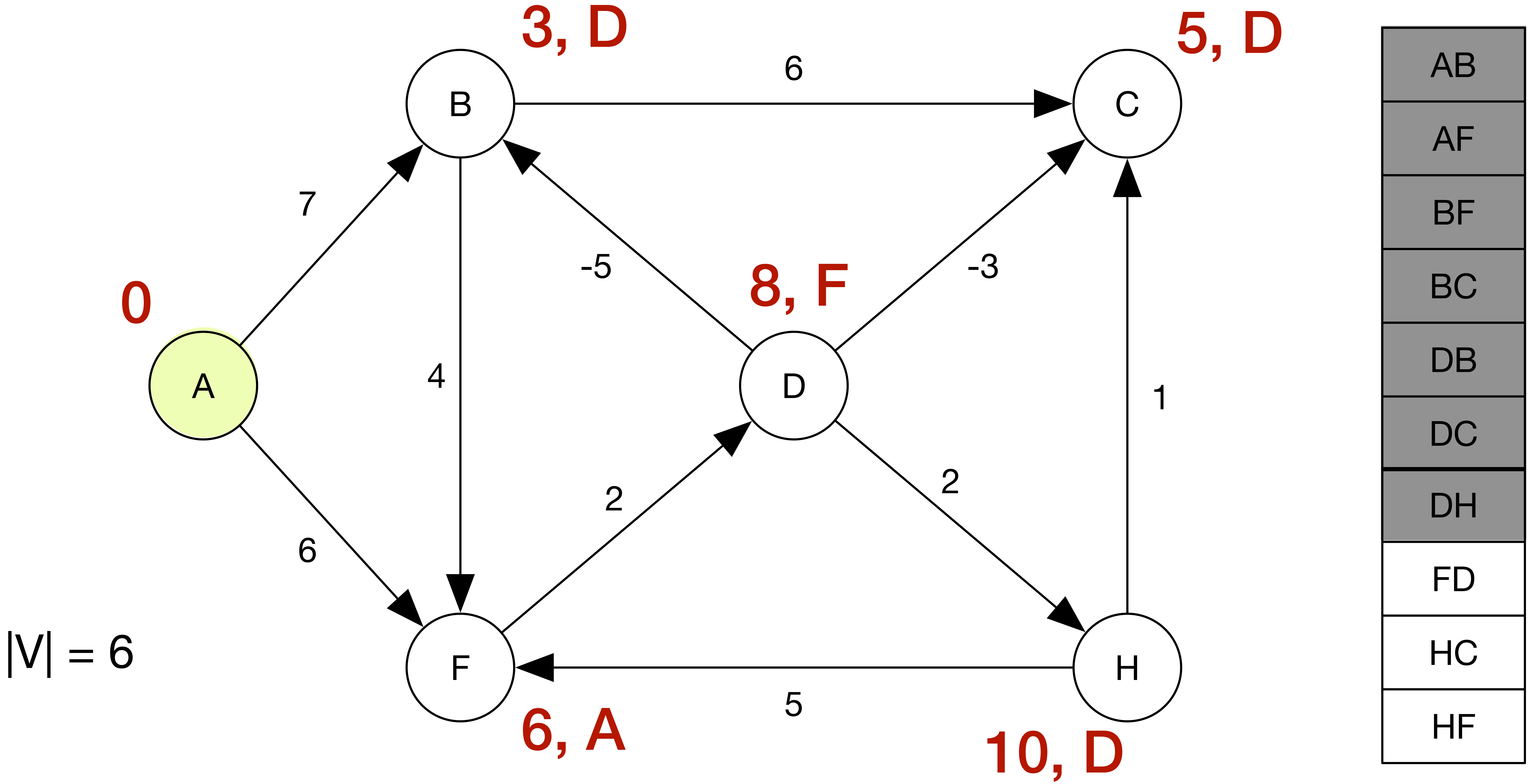

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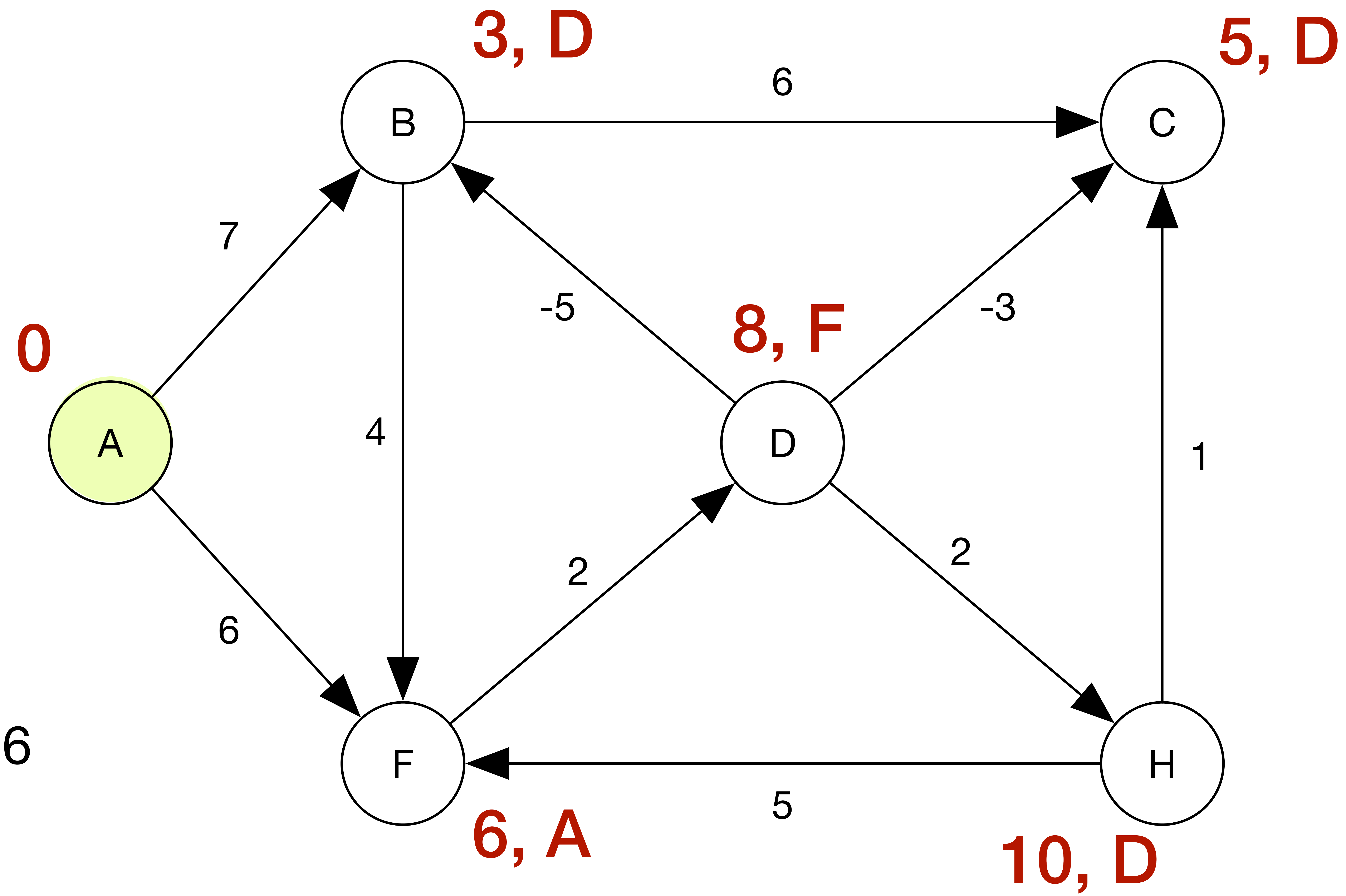

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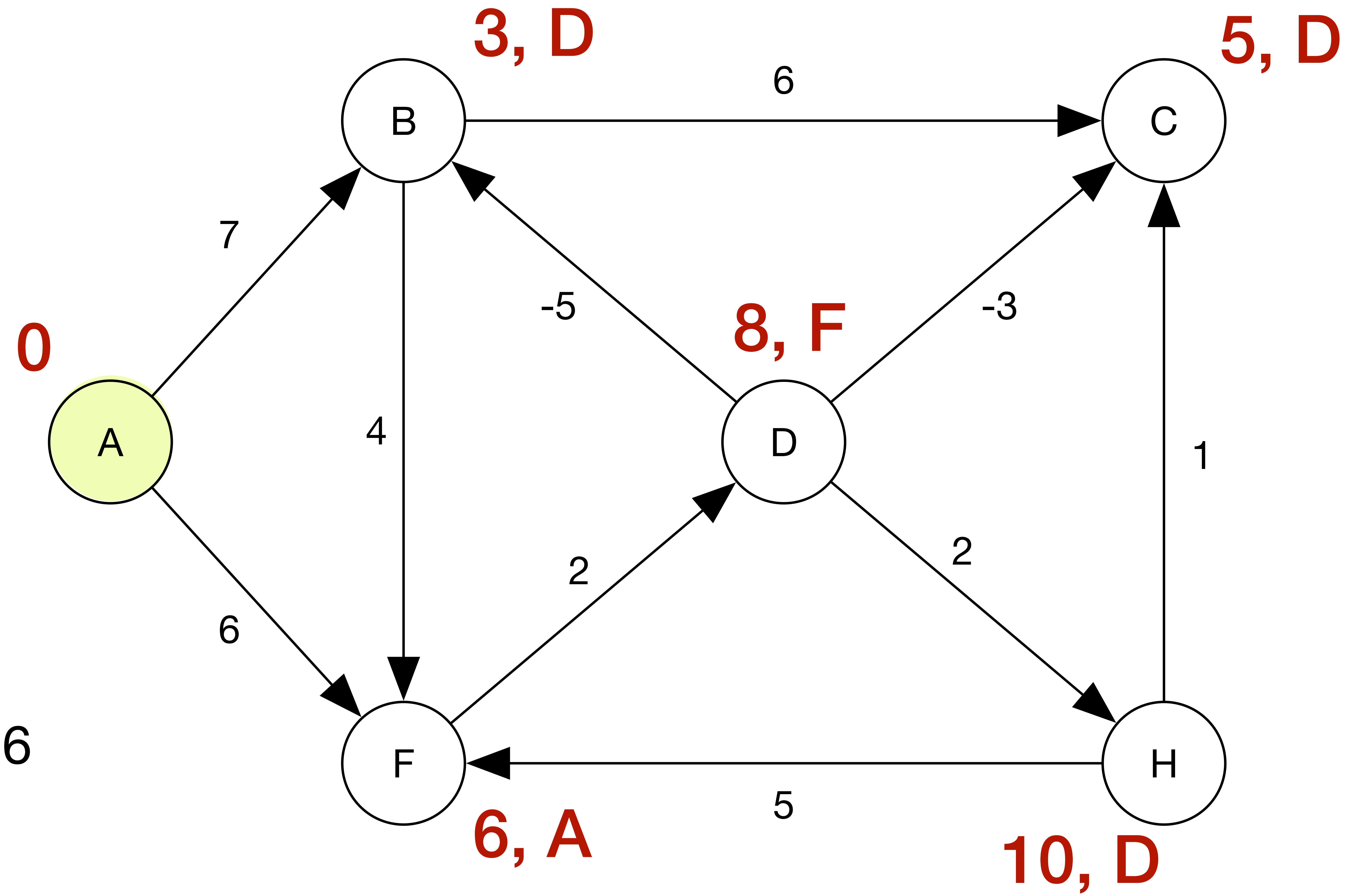

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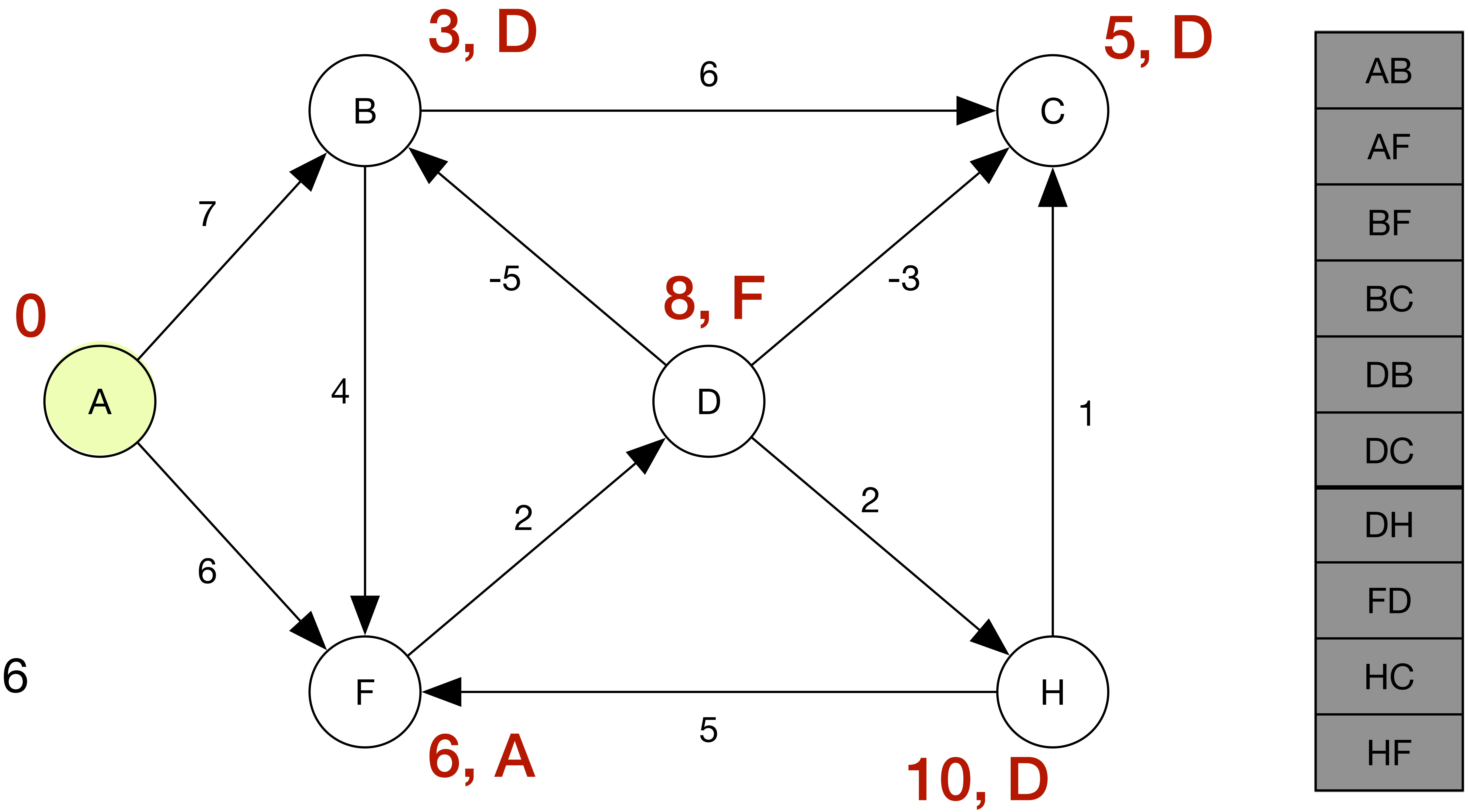

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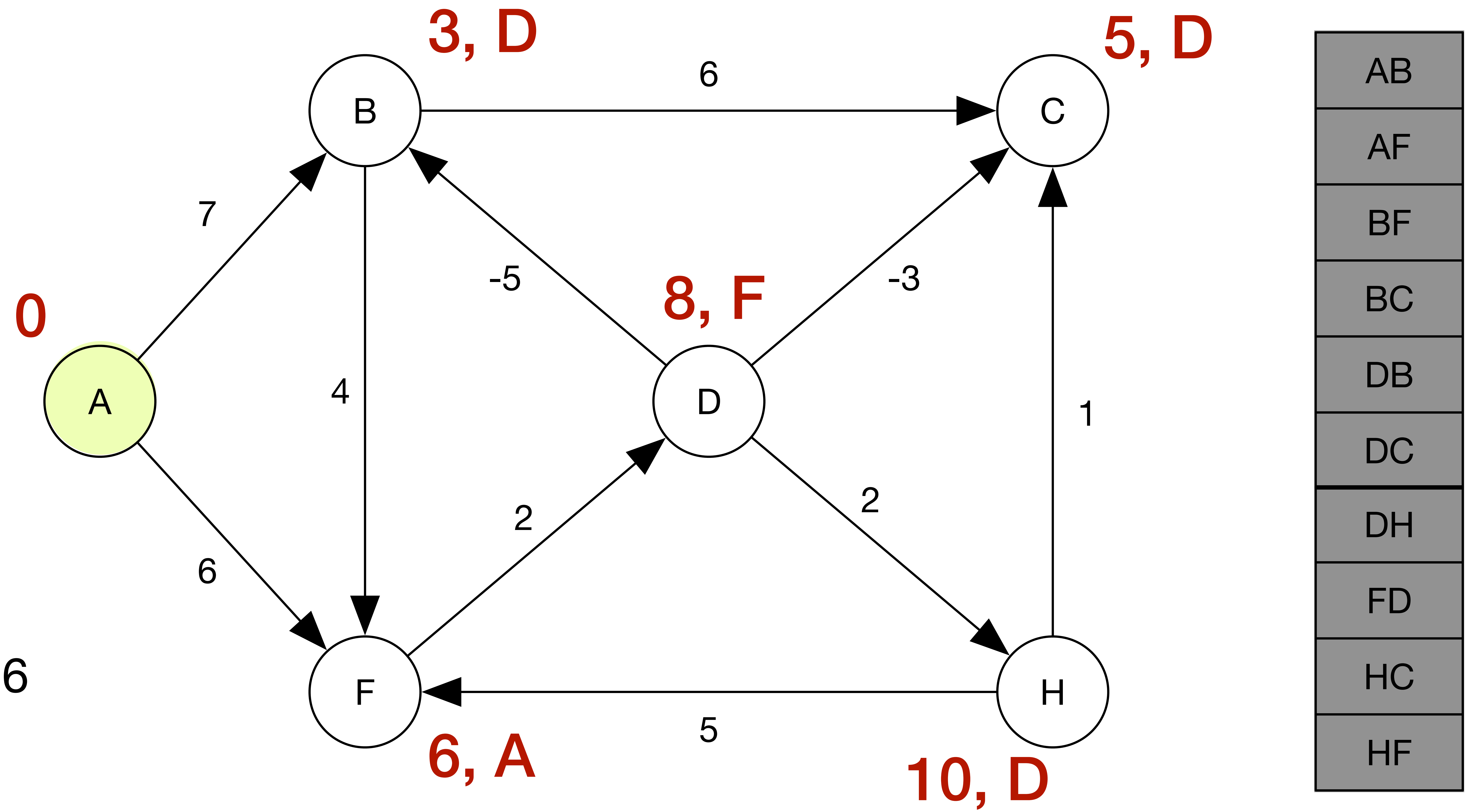
Bellman-Ford

Done!



|V| = 6

Bellman-Ford

Why do we need up to $|V| - 1$ iterations?

Bellman-Ford

Why do we need up to |V| - 1 iterations? If we have |V| nodes, then the shortest path can have no more than |V| - 1 edges. When we iterate, at the *k*th iteration, we know we've covered all shortest paths up to length *k*. To consider all possible shortest paths, we need $k = |V| - 1$.

Bellman-Ford

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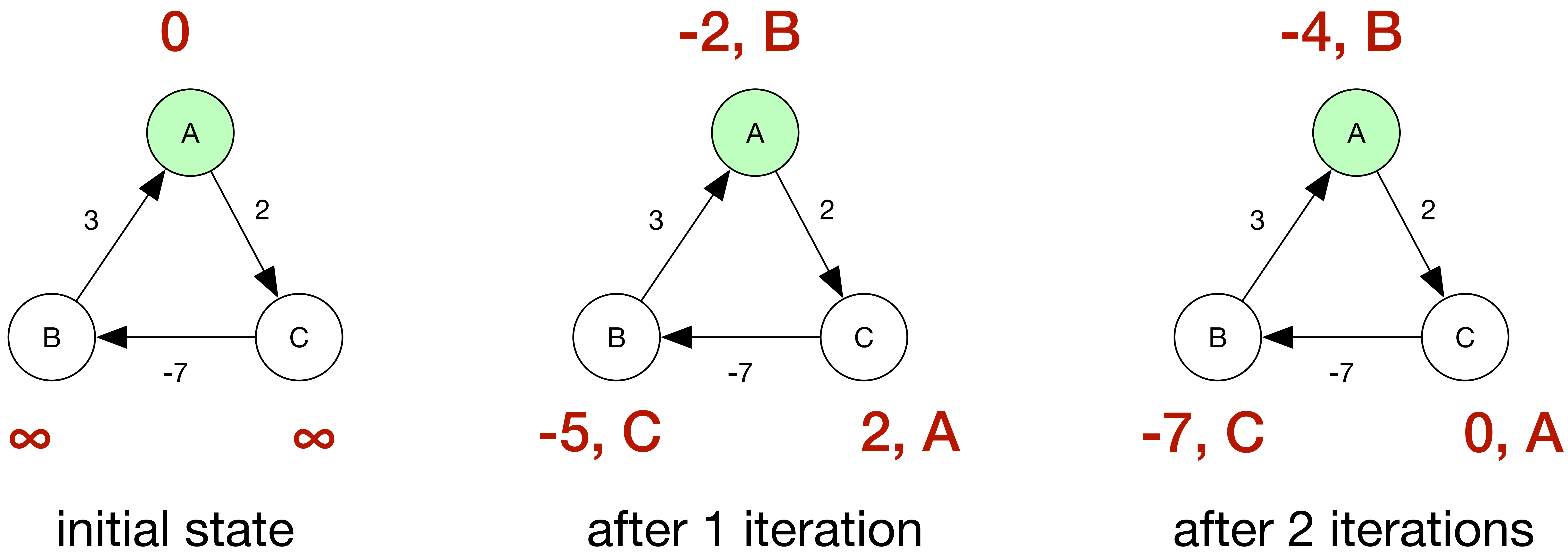
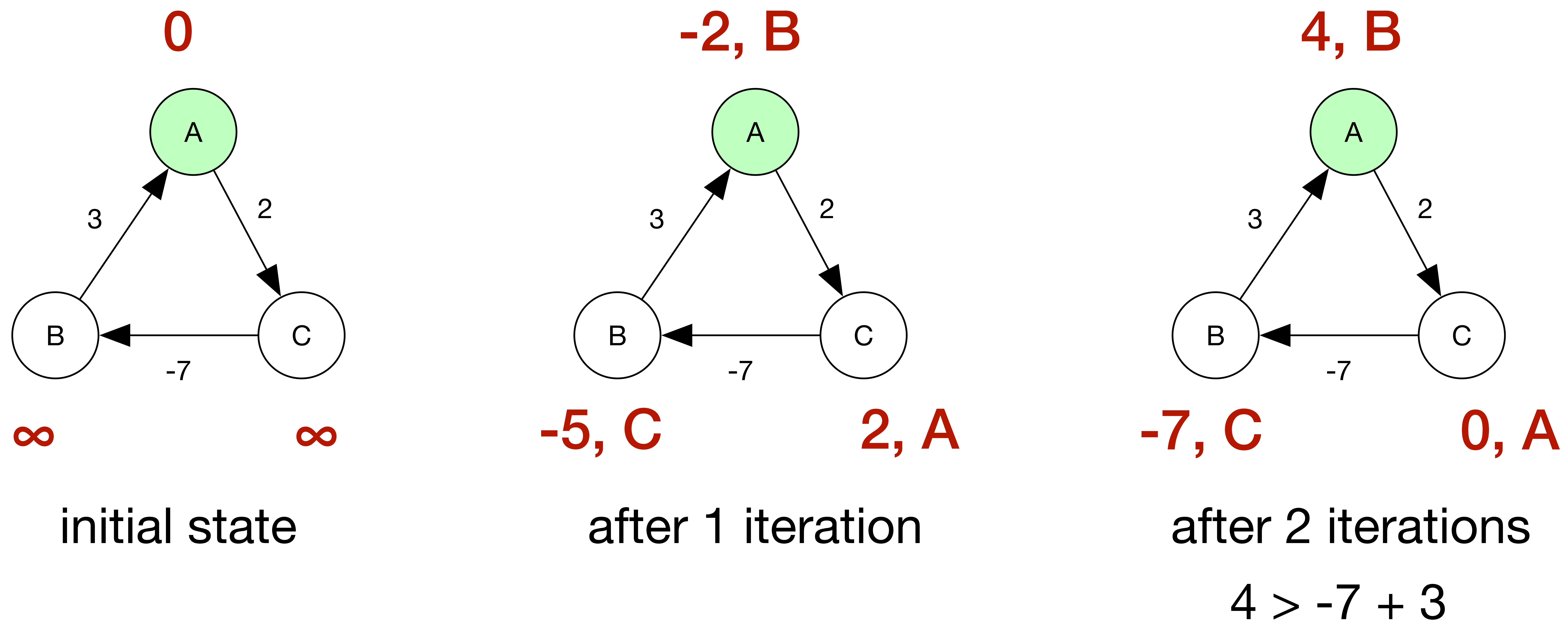
Negative cycle detection



Bellman-Ford

Negative cycle detection

```
for edge (U, V),
if distance[V] > distance[U] + weight of edge
then we have a negative cycle
```



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Shortest Path

Algorithm	Dijkstra	Bellman - Ford
Worst-case complexity	O((|V| + |E|) log |V|)	O(|V| x |E|)
Restrictions	Edge weights must be non-negative	No negative cycles