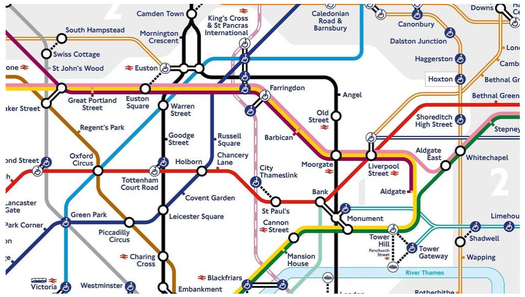
THE UNIVERSITY OF VERMONT
COLLEGE OF ENGINEERING &
MATHEMATICAL SCIENCES

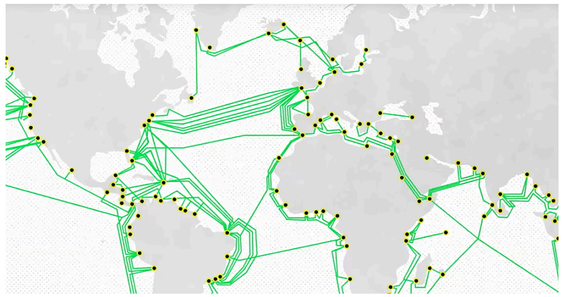
Network Flows

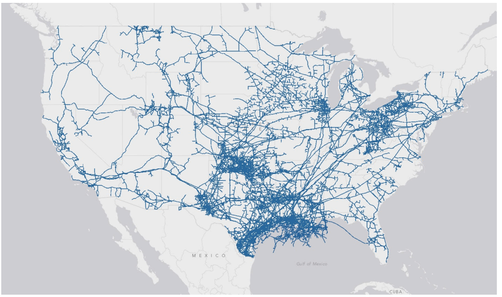
Max flow / min cut: Ford-Fulkerson, part two

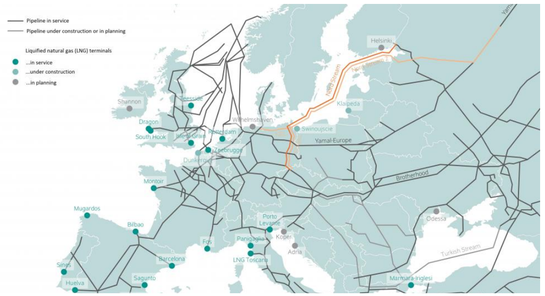
CS 124 / Department of Computer Science

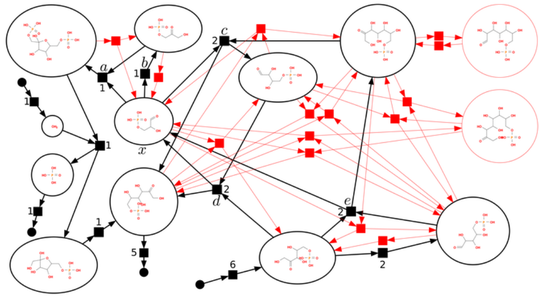
Network flow





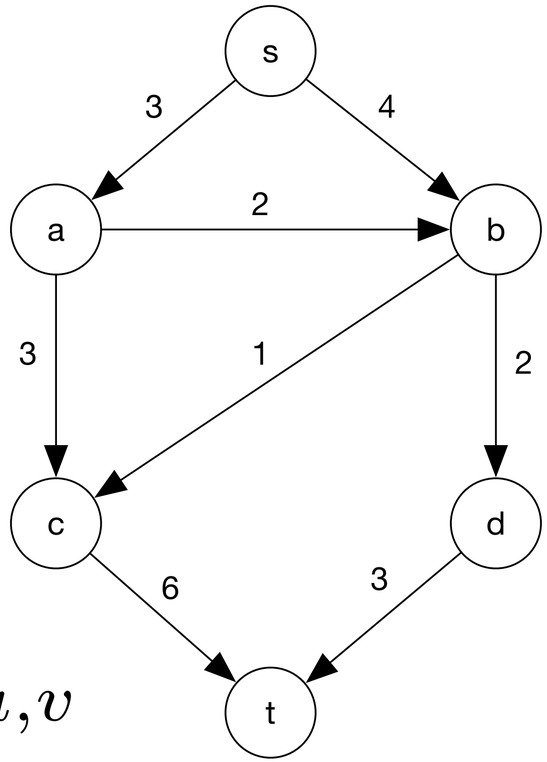






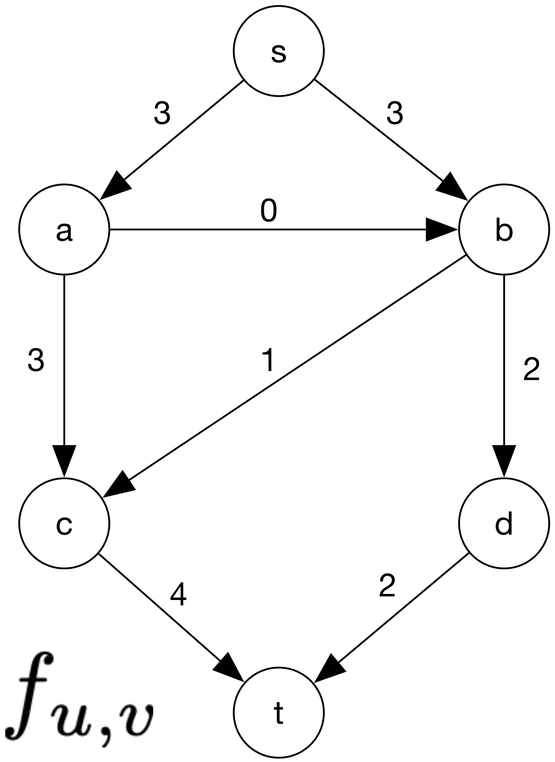
Network flow

CAPACITY



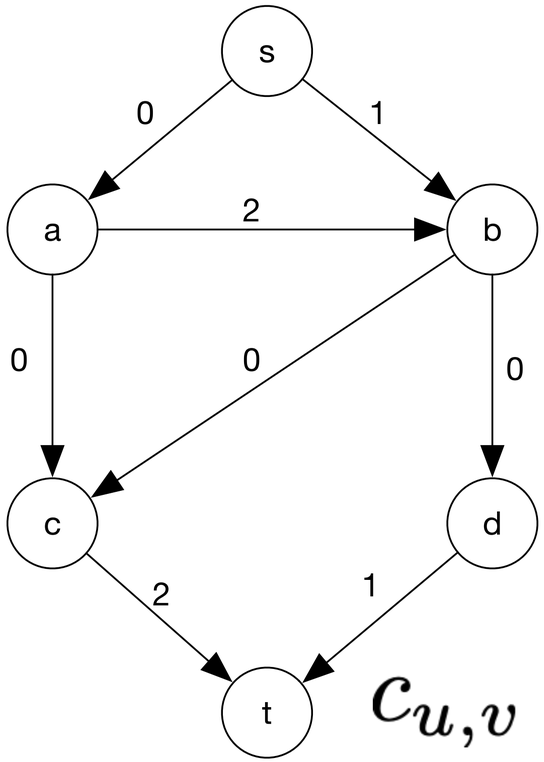
$c_{u,v}$

FLOW



$f_{u,v}$

RESIDUAL



$c_{u,v} - f_{u,v}$

Network flow

$f_{u,v}$ is the flow from u to v

$$f_{u,v} \leq c_{u,v}$$

$$\sum_{u:(u,v)\in E} f_{u,v} = \sum_{w:(v,w)\in E} f_{v,w}$$

$$\sum_{u:(s,u)\in E} = \sum_{w:(w,t)\in E}$$

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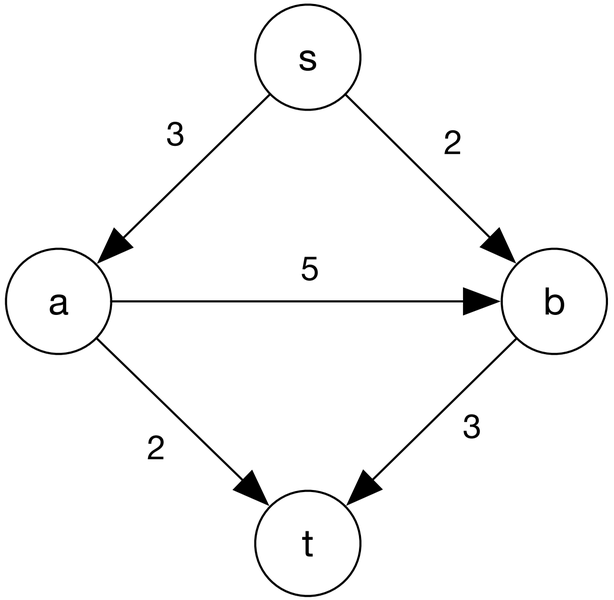
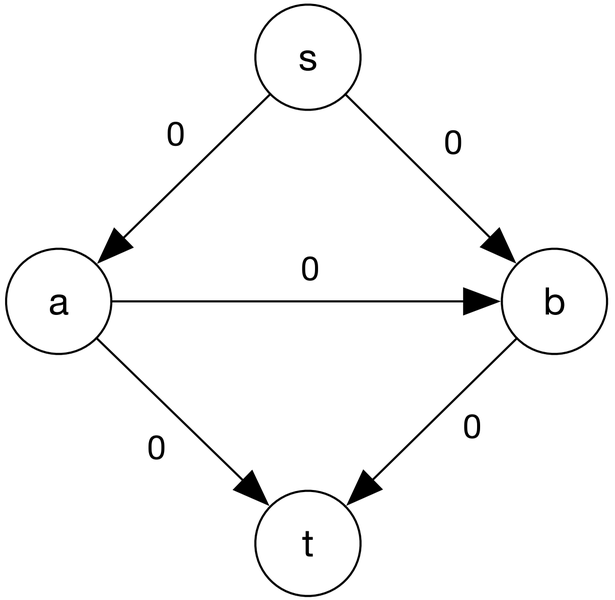


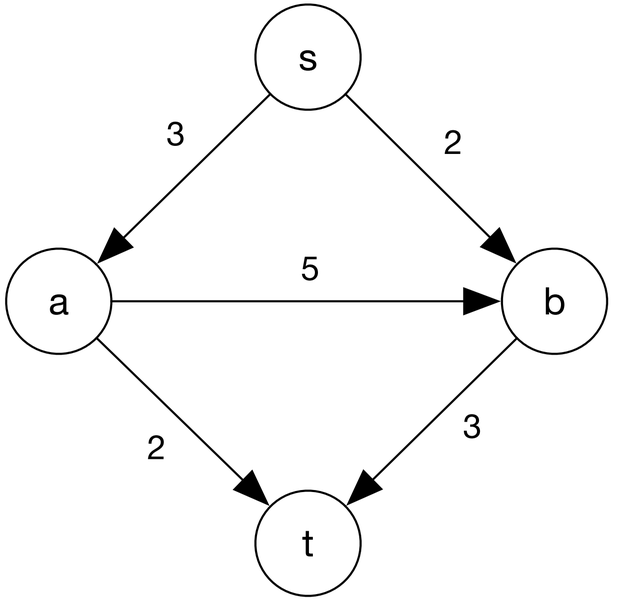
$c_{u,v}$

FLOW



$f_{u,v}$

RESIDUAL



$c_{u,v} - f_{u,v}$

Network flow

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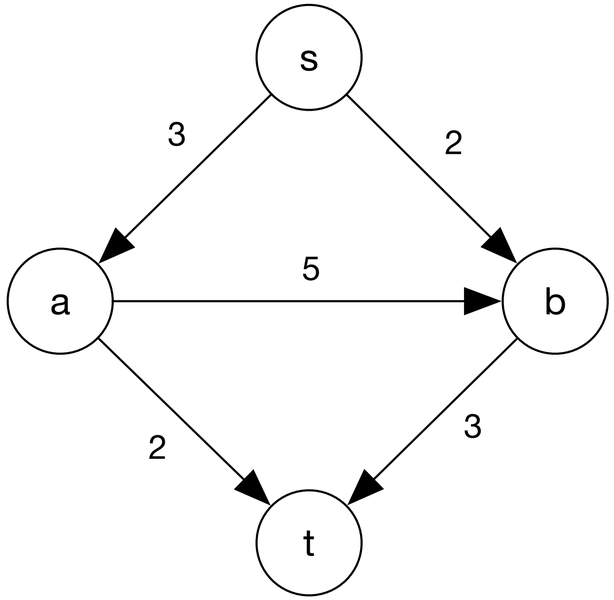

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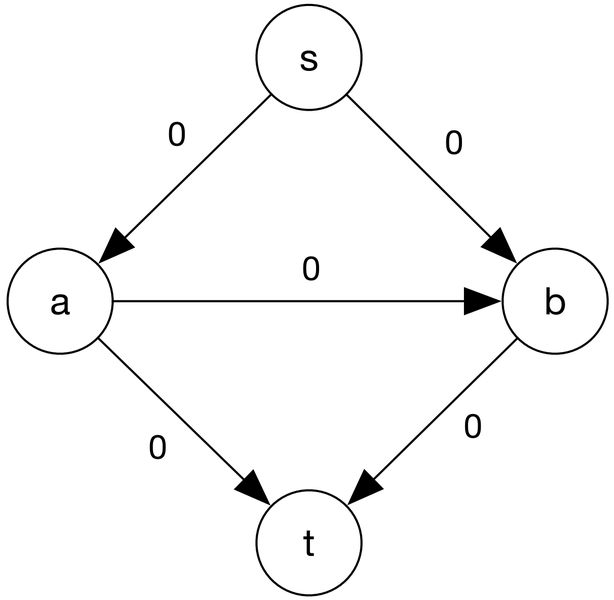

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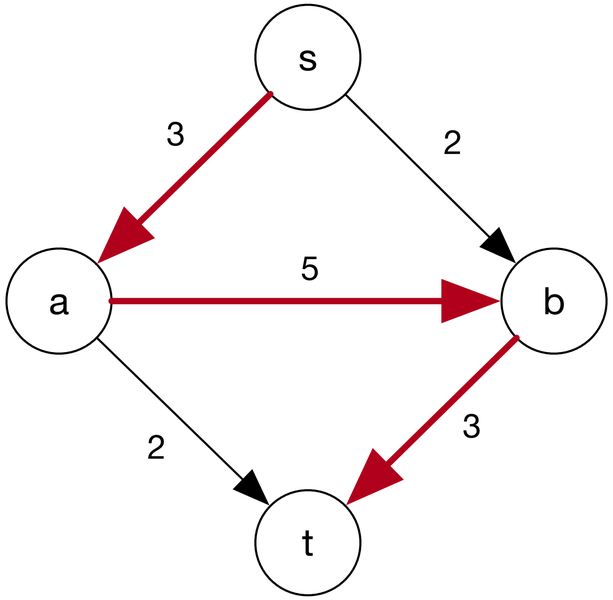

Network flow

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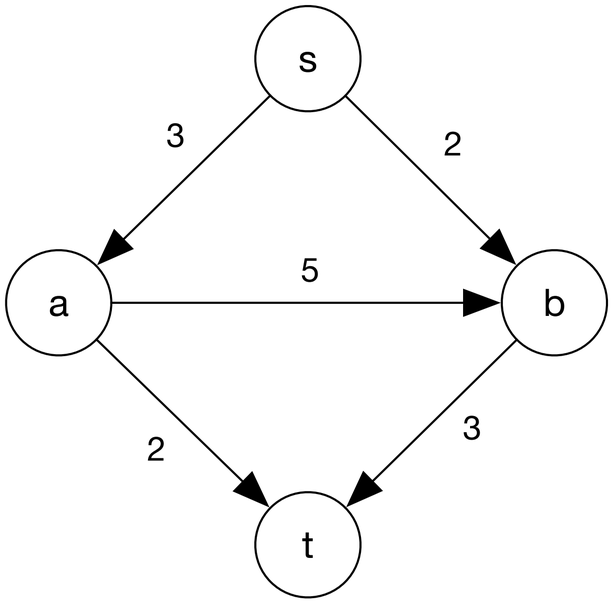

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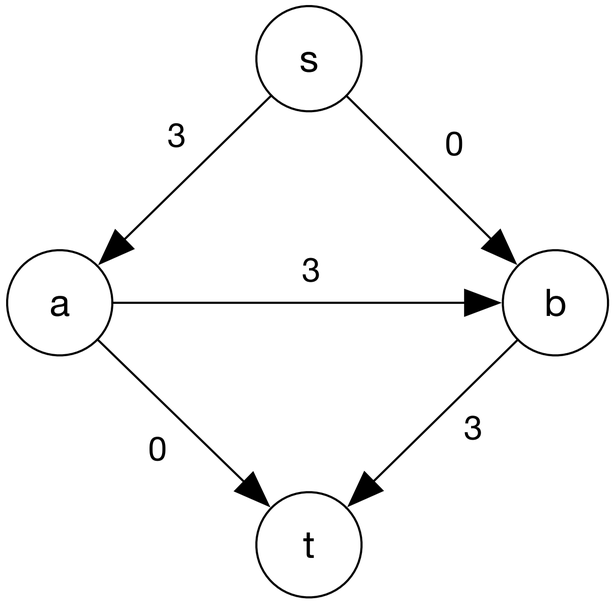

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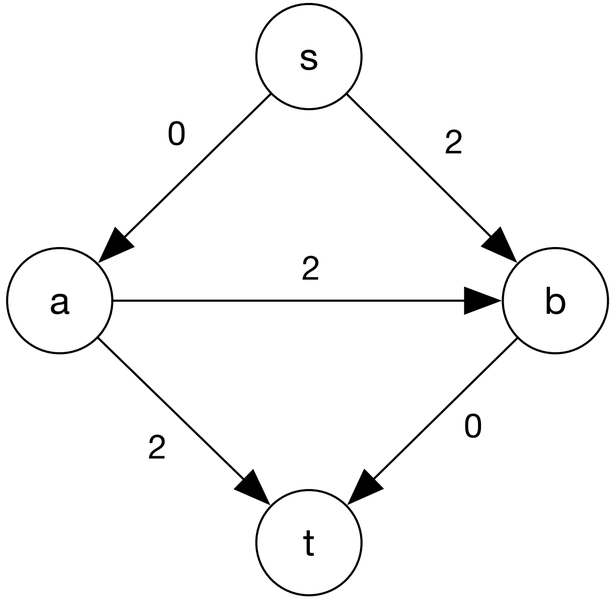
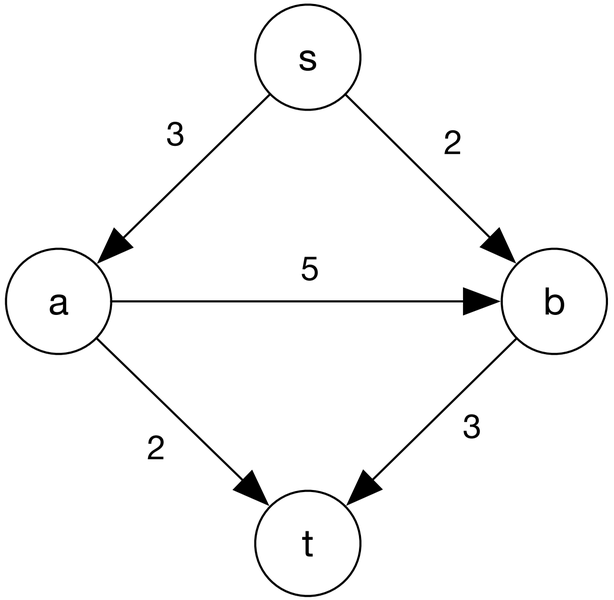


We have a problem!

Network flow

Start over!

CAPACITY



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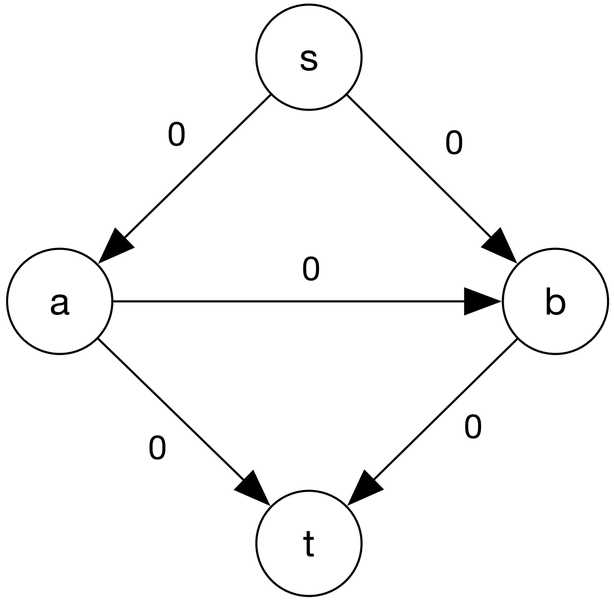

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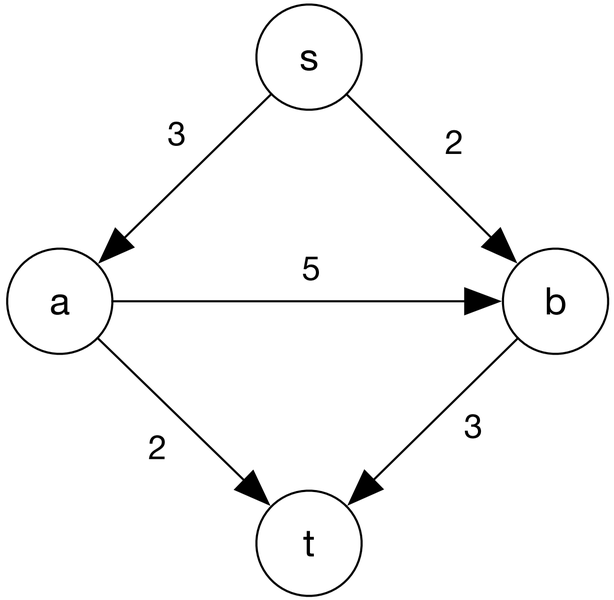
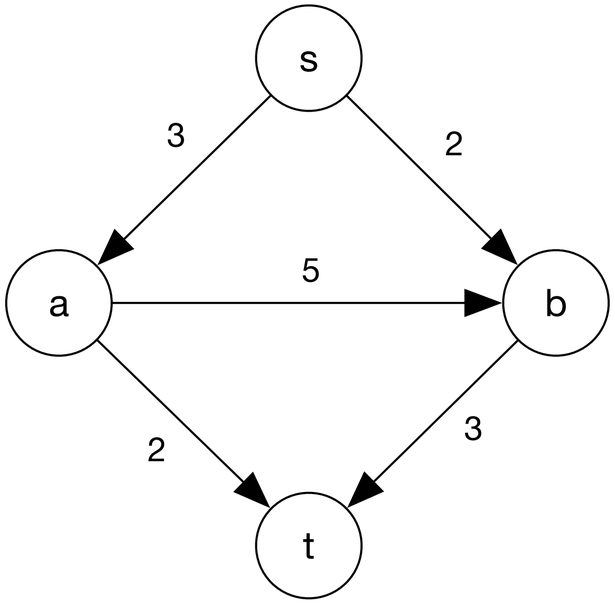


Network flow

CAPACITY



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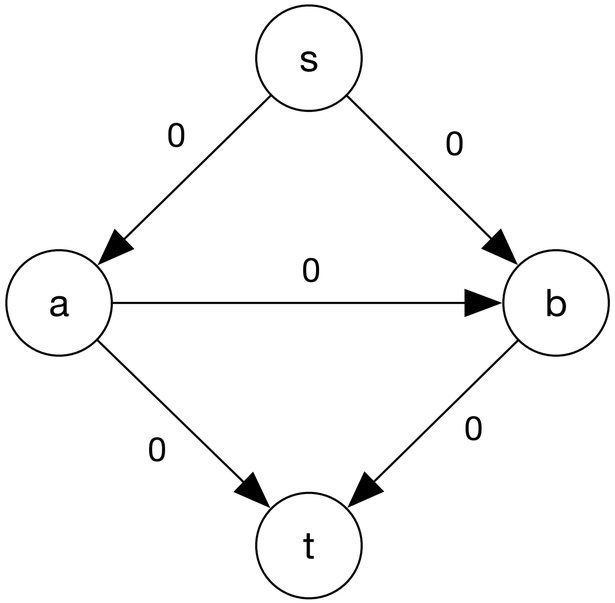

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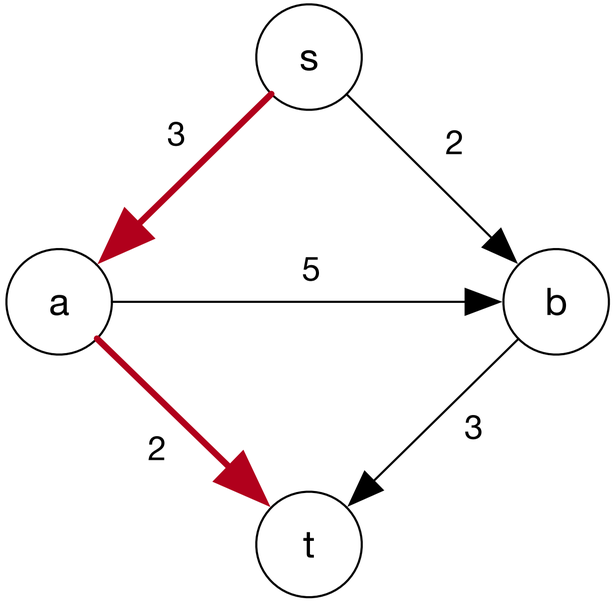
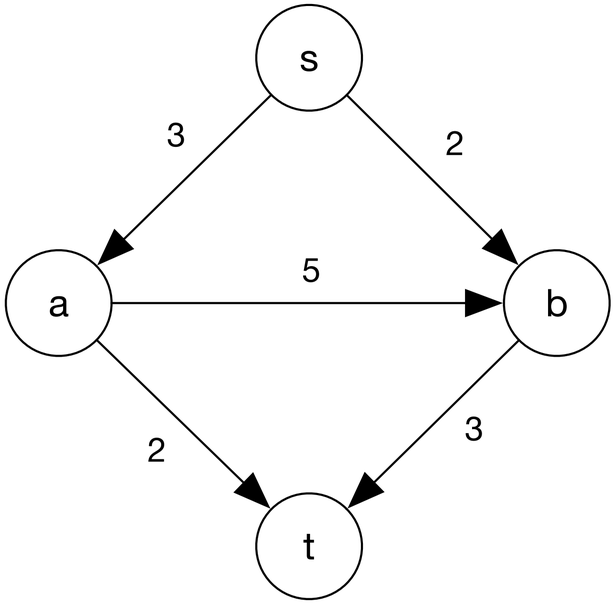


Network flow

CAPACITY



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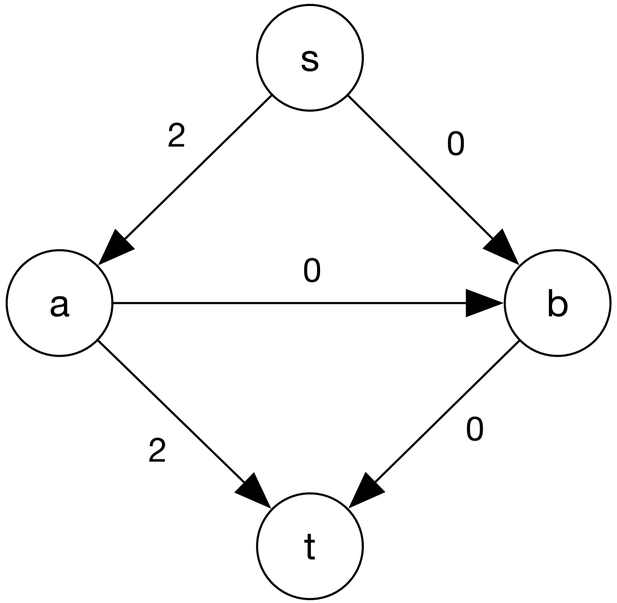

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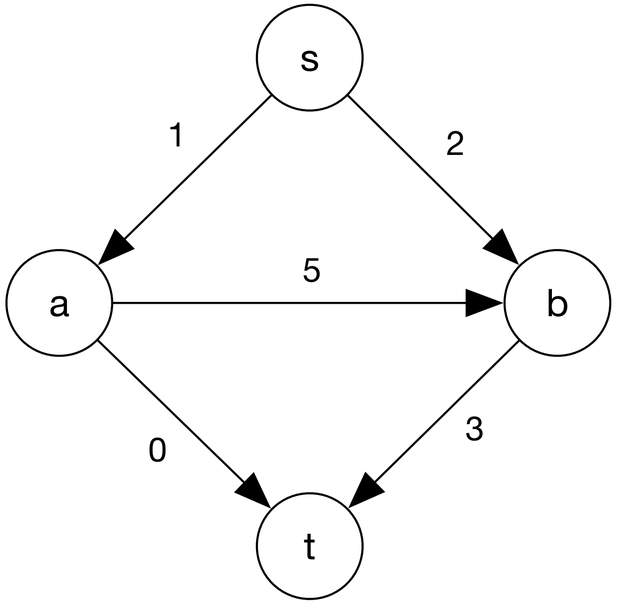
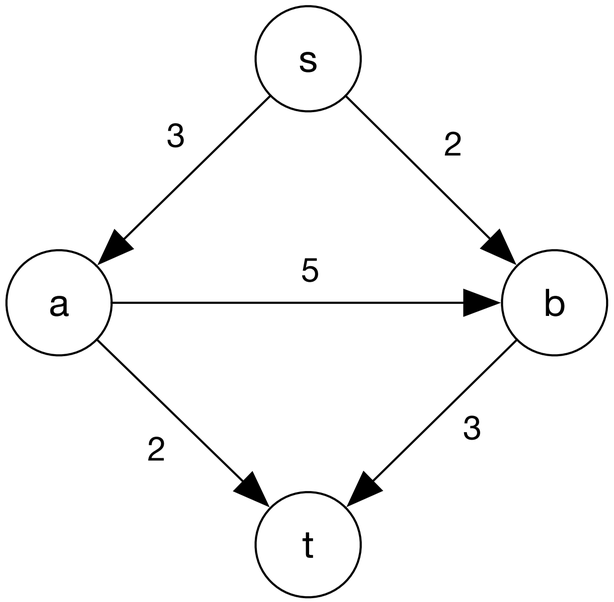


Network flow

CAPACITY



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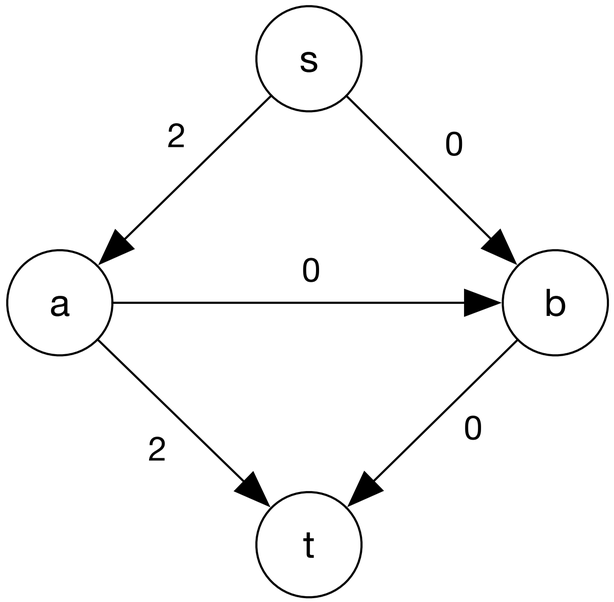

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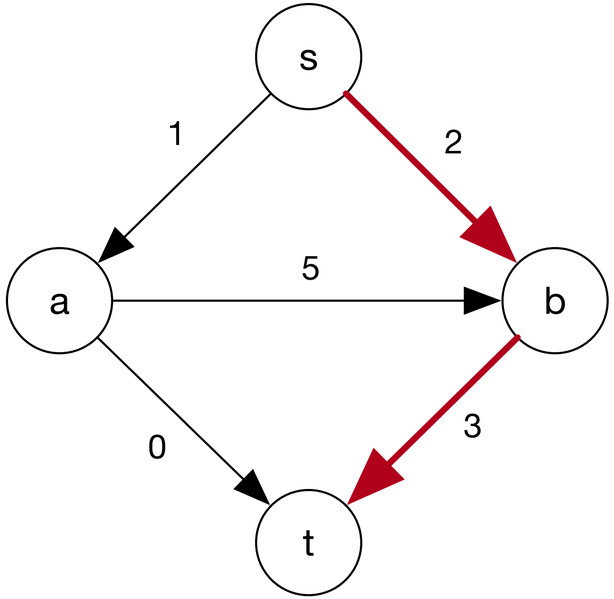

Network flow

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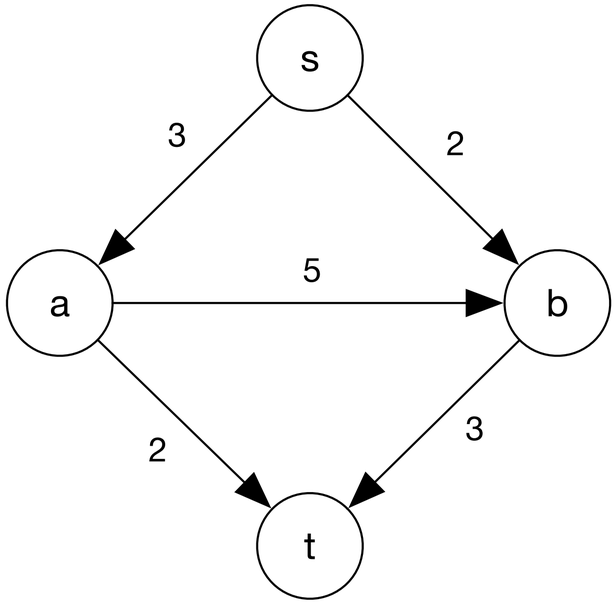

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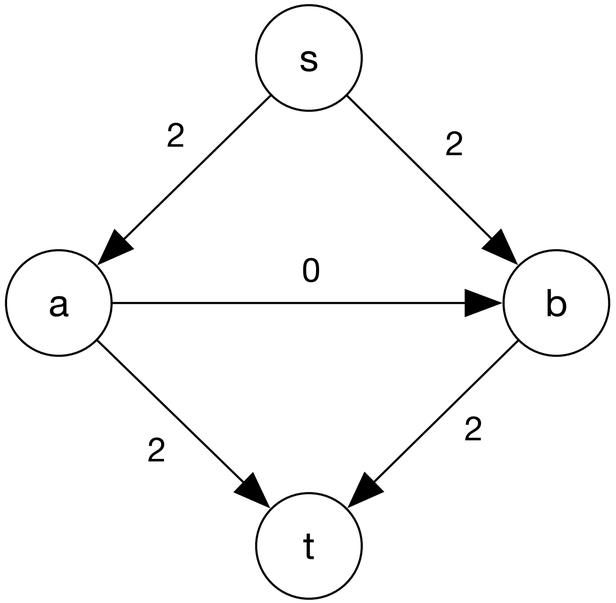

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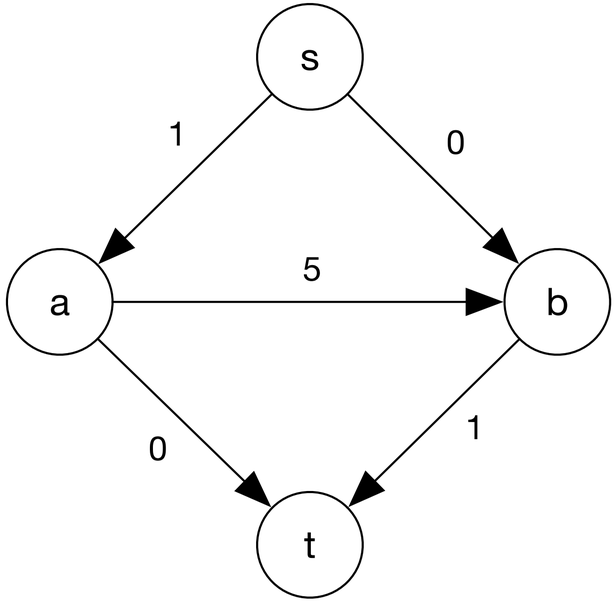

Network flow

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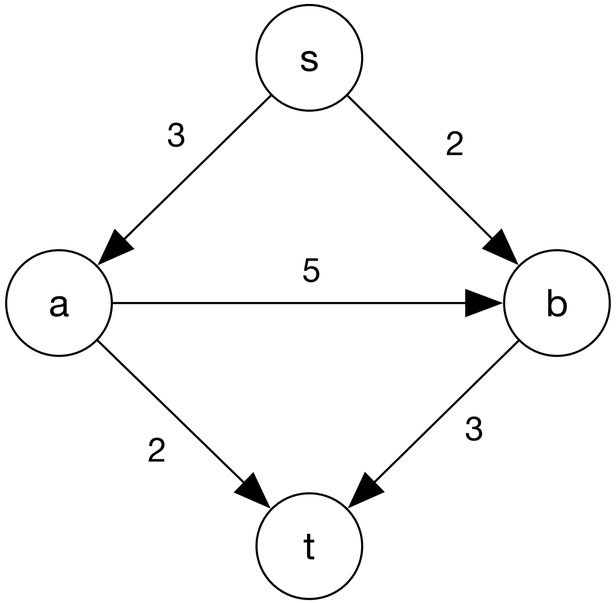

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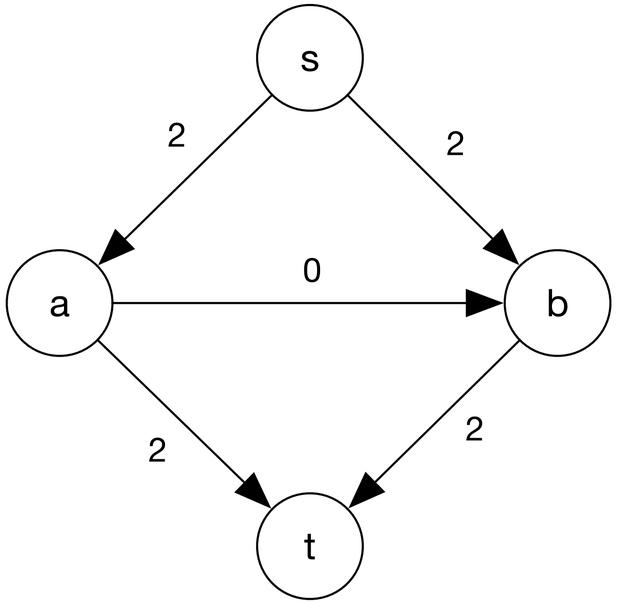

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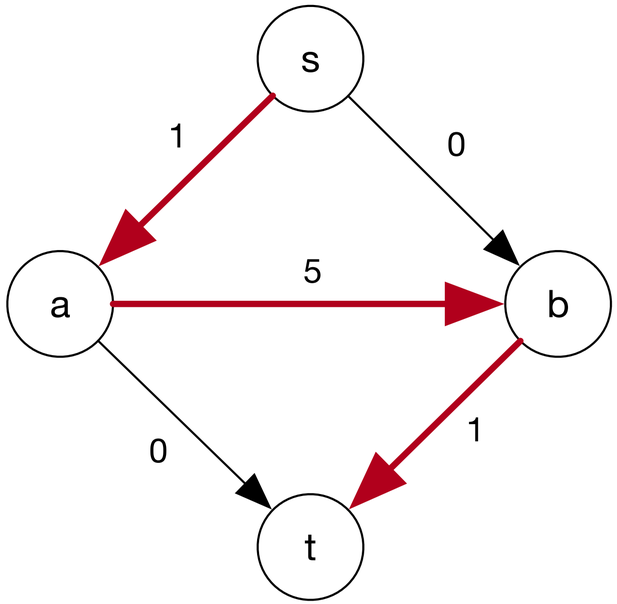
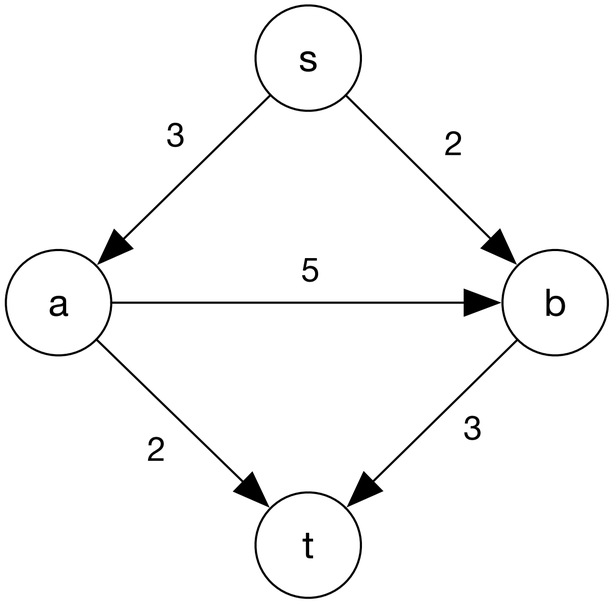


Network flow

CAPACITY



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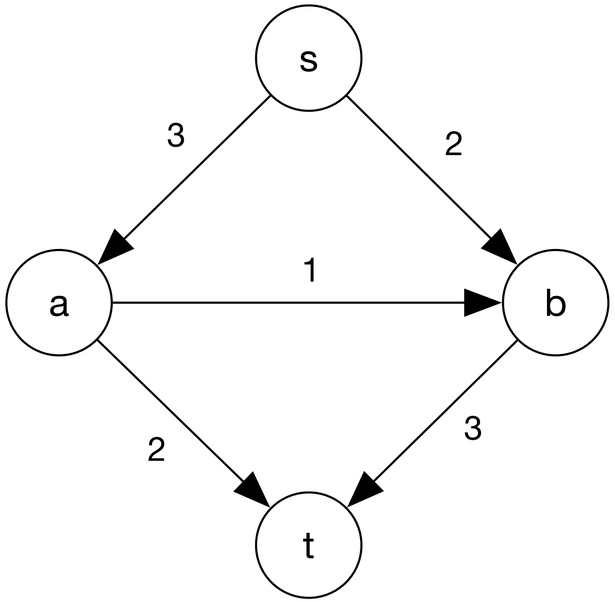

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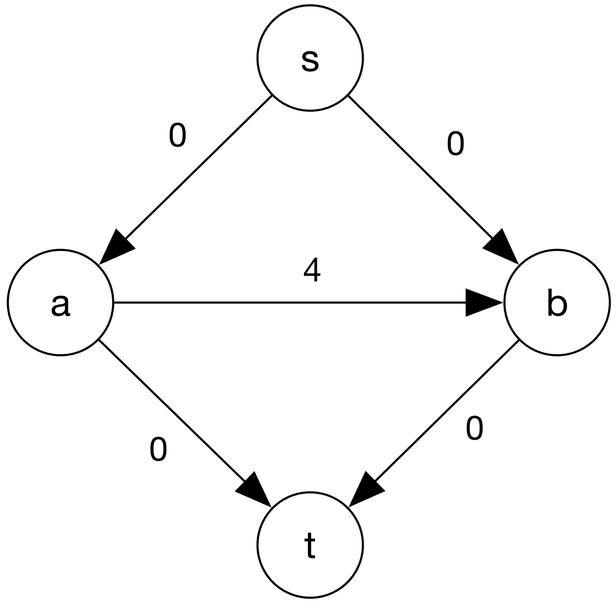


Now we have found the maximum flow

Network flow

Problem: By a "bad" sequence of discoveries of augmenting paths, we might discover a "blocking flow" which prevents us from discovering other augmenting paths.

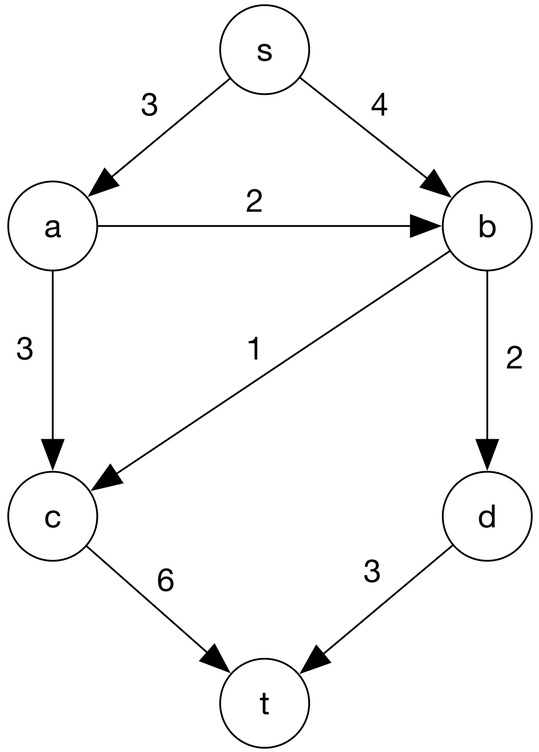
Network flow

Problem: By a "bad" sequence of discoveries of augmenting paths, we might discover a "blocking flow" which prevents us from discovering other augmenting paths.

Solution: Encode a way of revising flows in the residual graph: add "backward" edges representing flow, in addition to "forward" edges representing unused capacity.

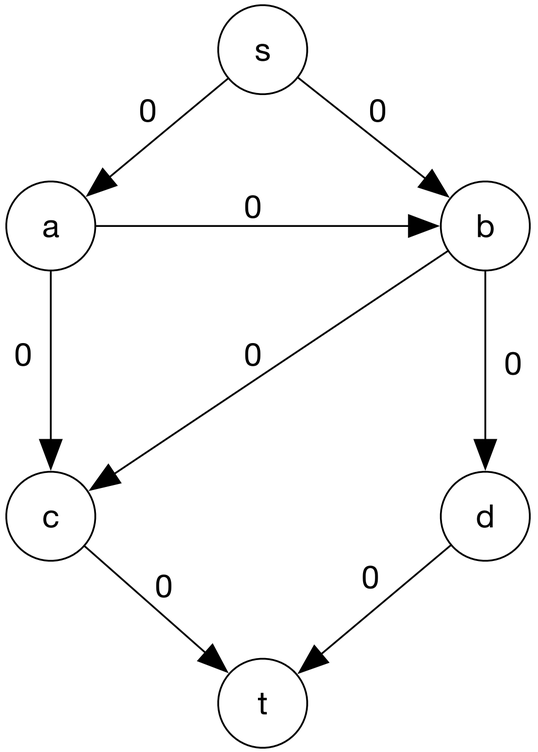
Network flow

CAPACITY



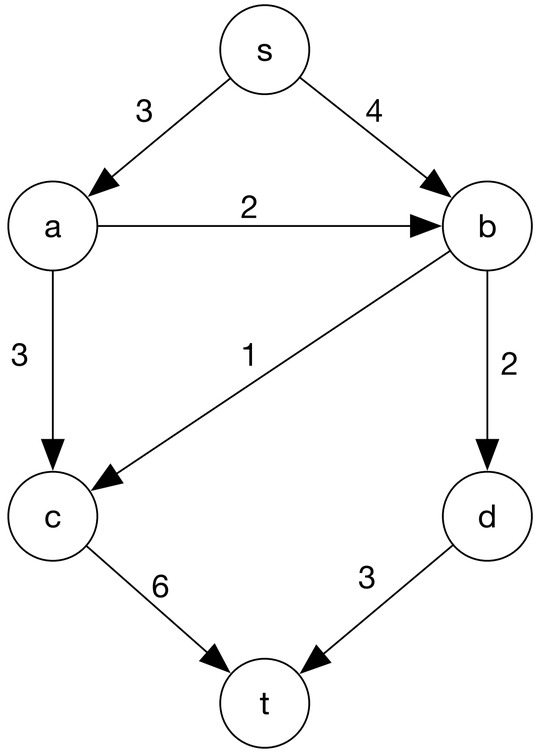
$c_{u,v}$

FLOW



$f_{u,v}$

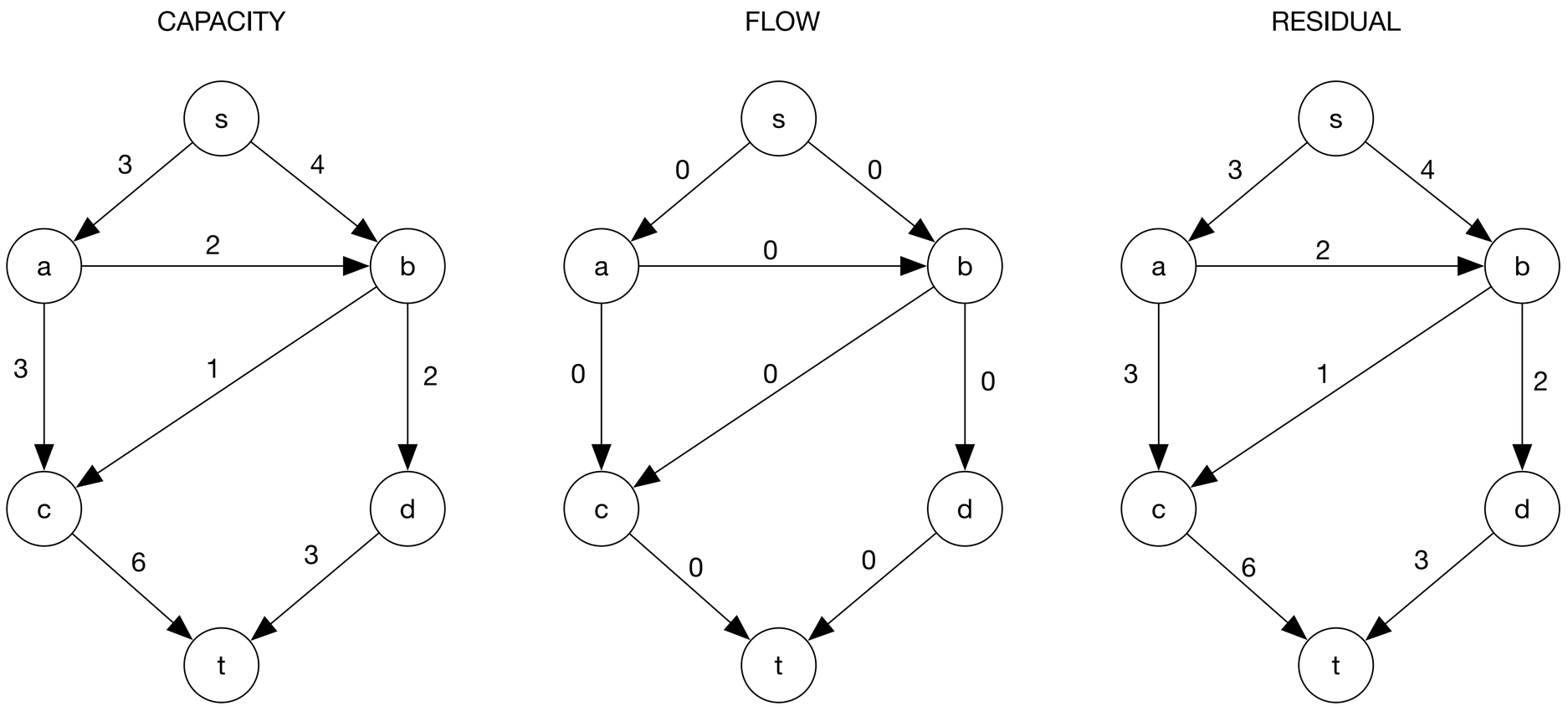
RESIDUAL



$c_{u,v} - f_{u,v}$

Network flow

Network flow



Network flow

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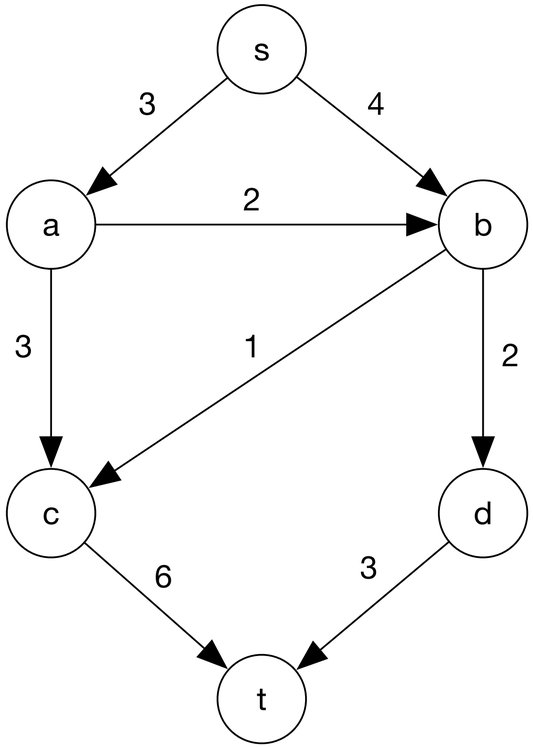

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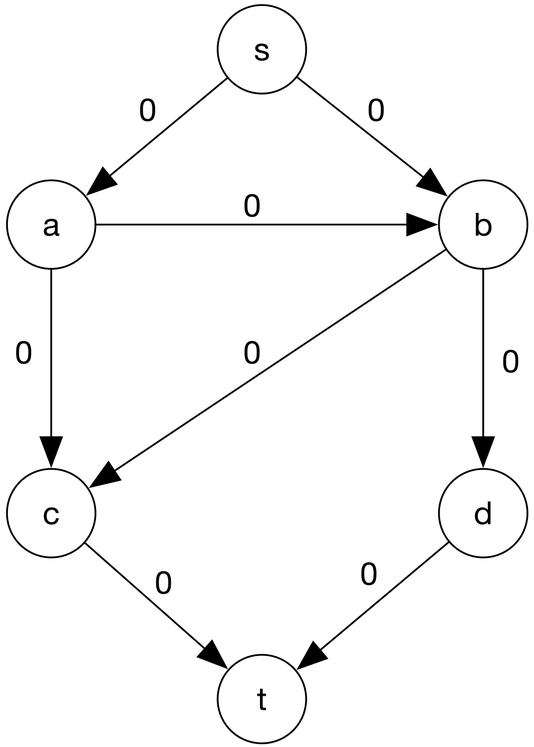

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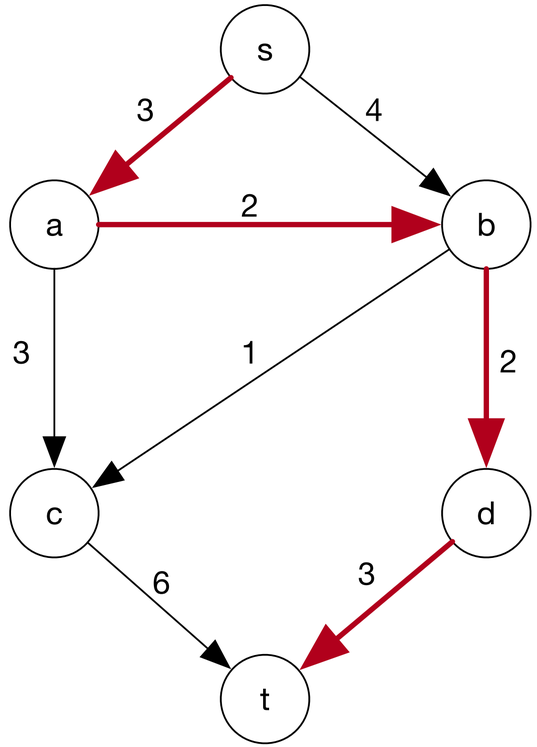
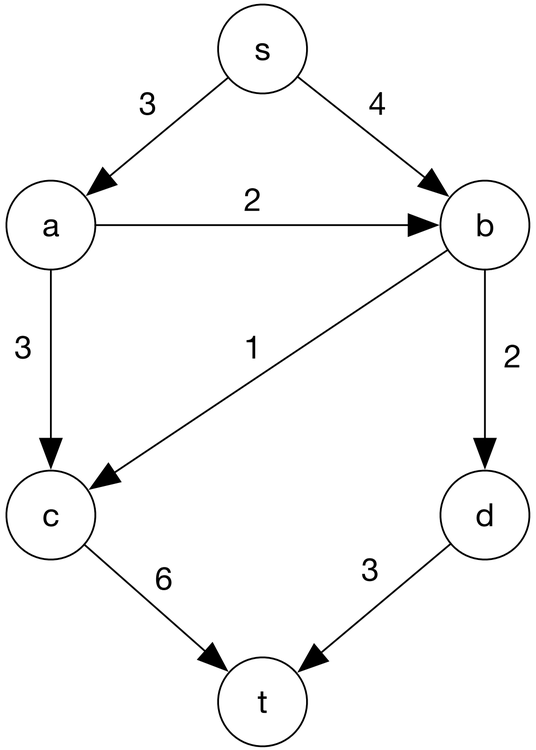


Network flow

CAPACITY



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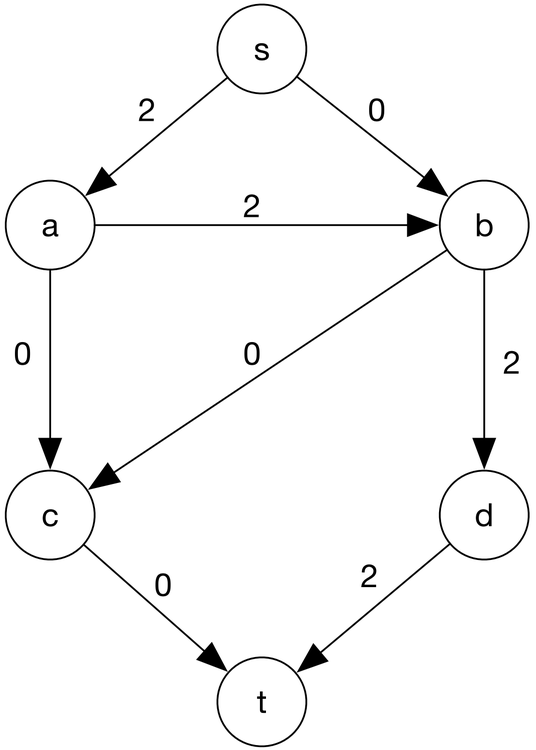

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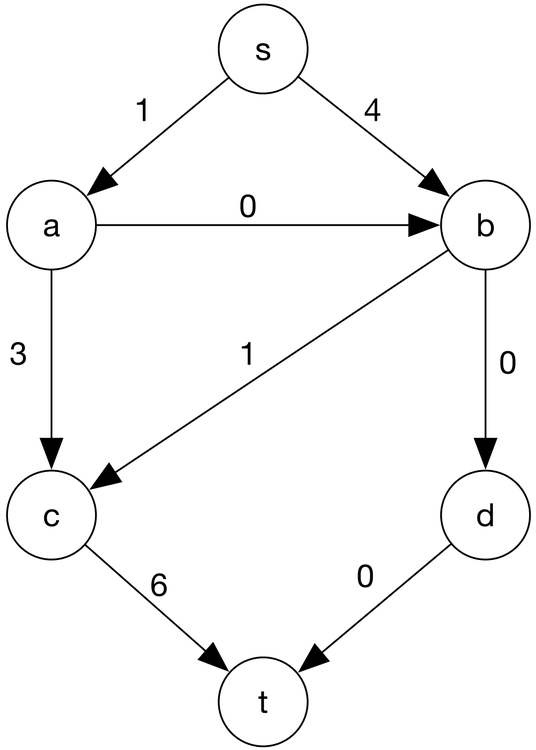
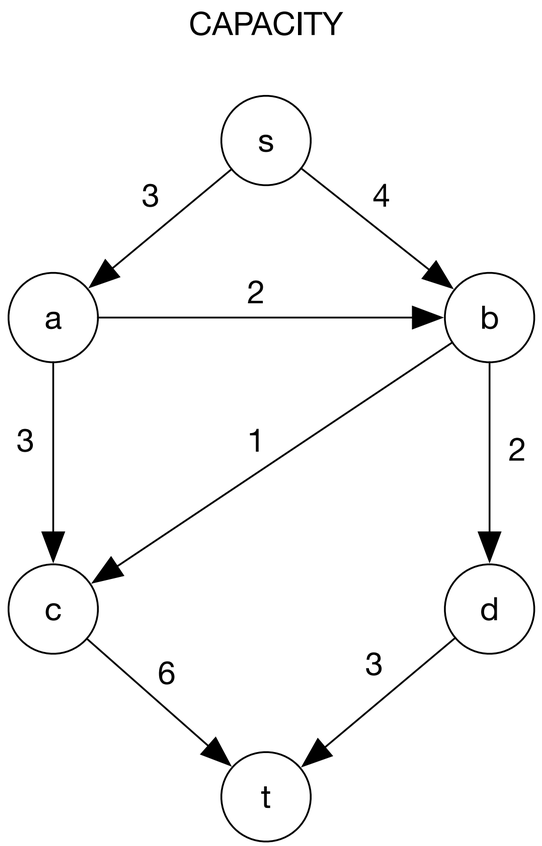
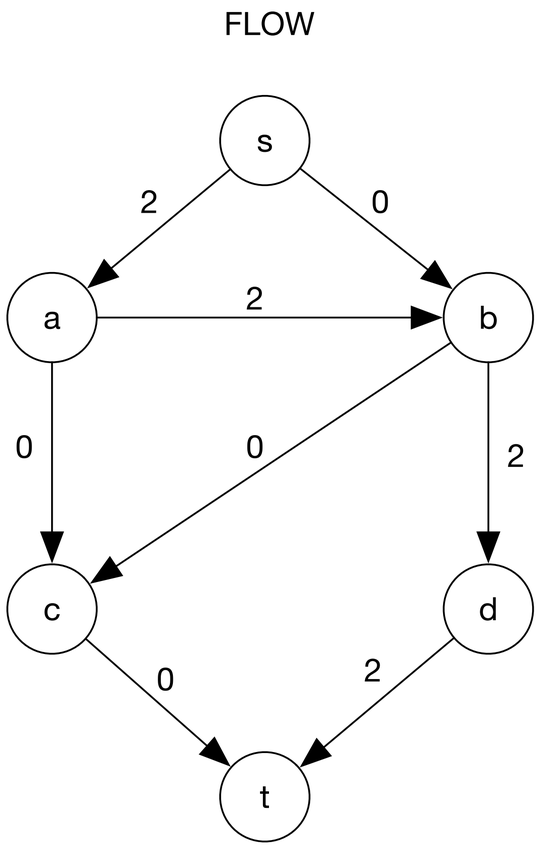
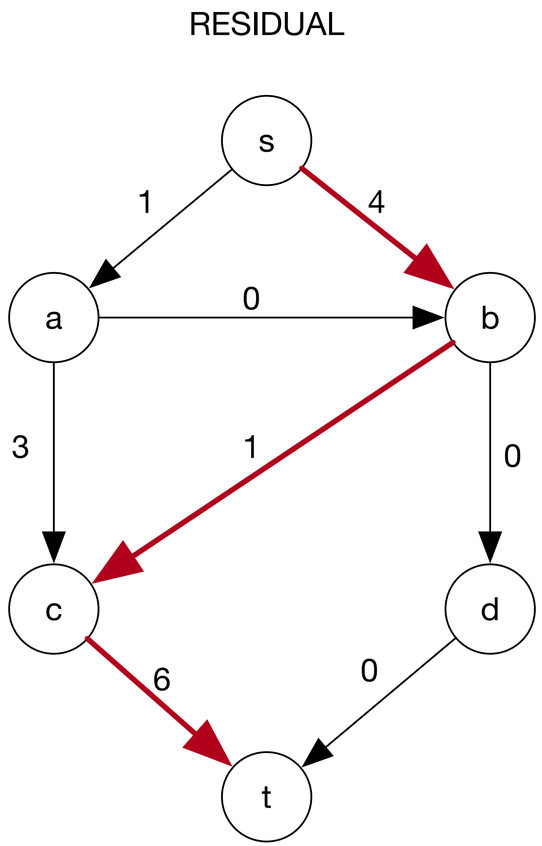


Network flow







Network flow

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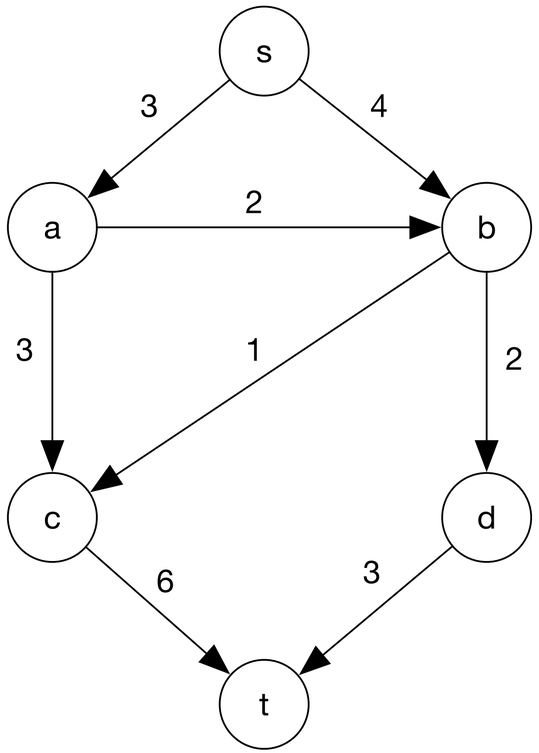

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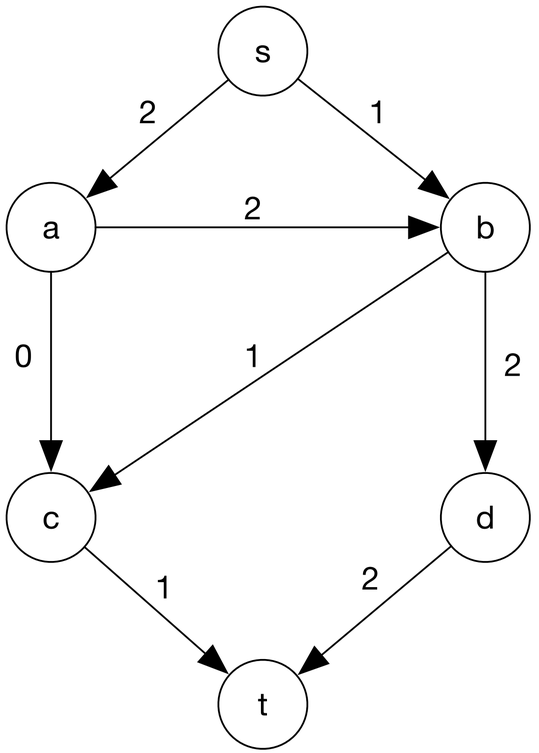

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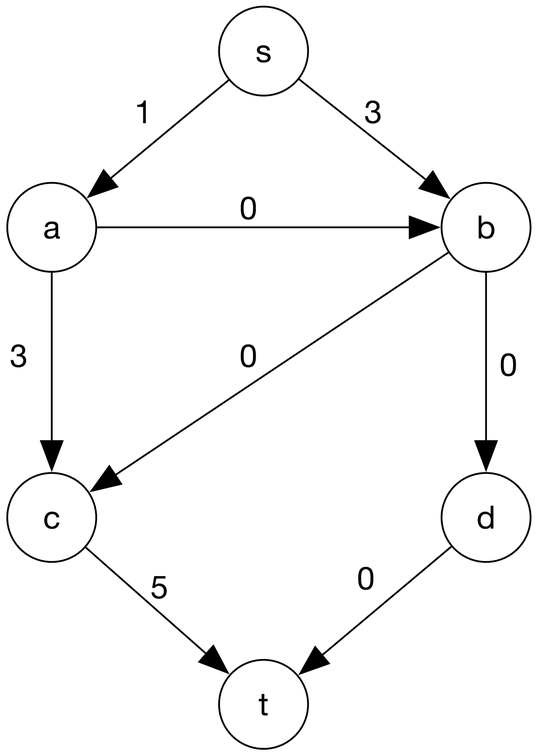

Network flow

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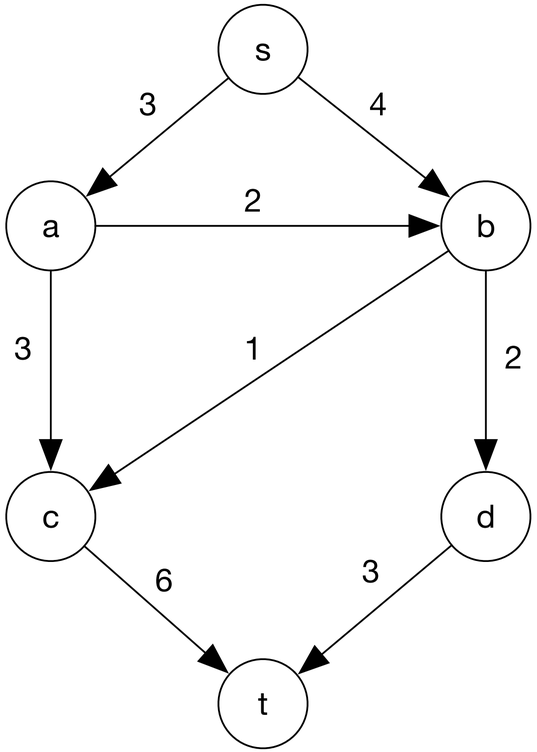
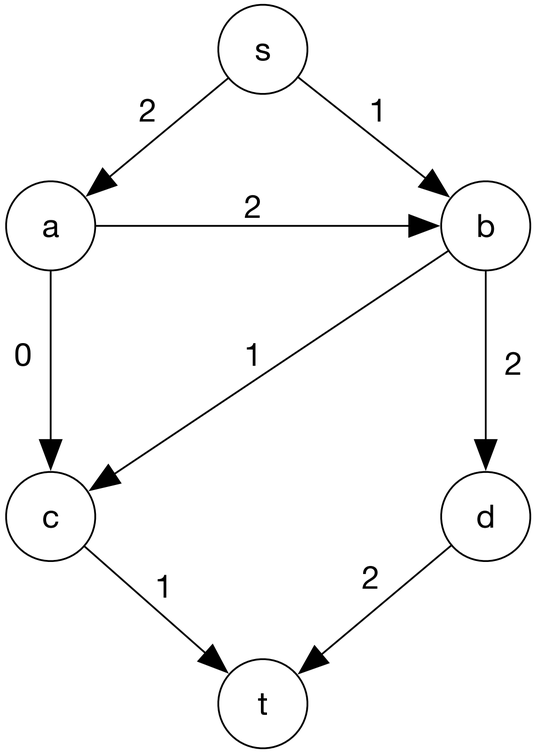
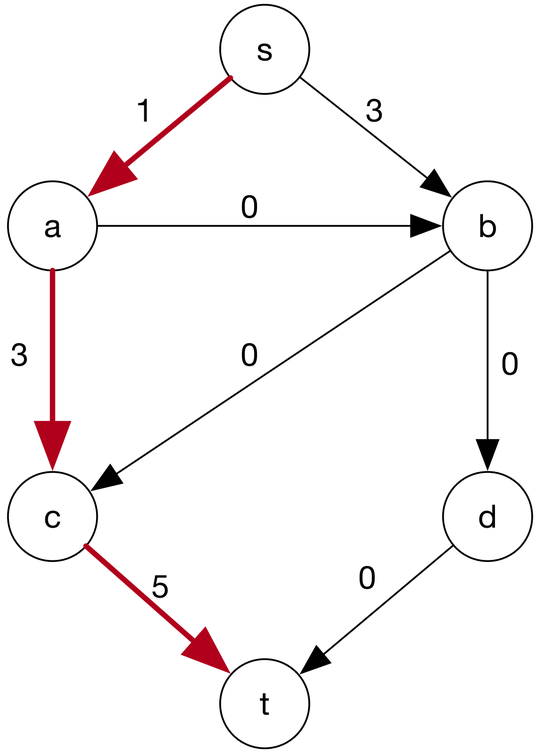


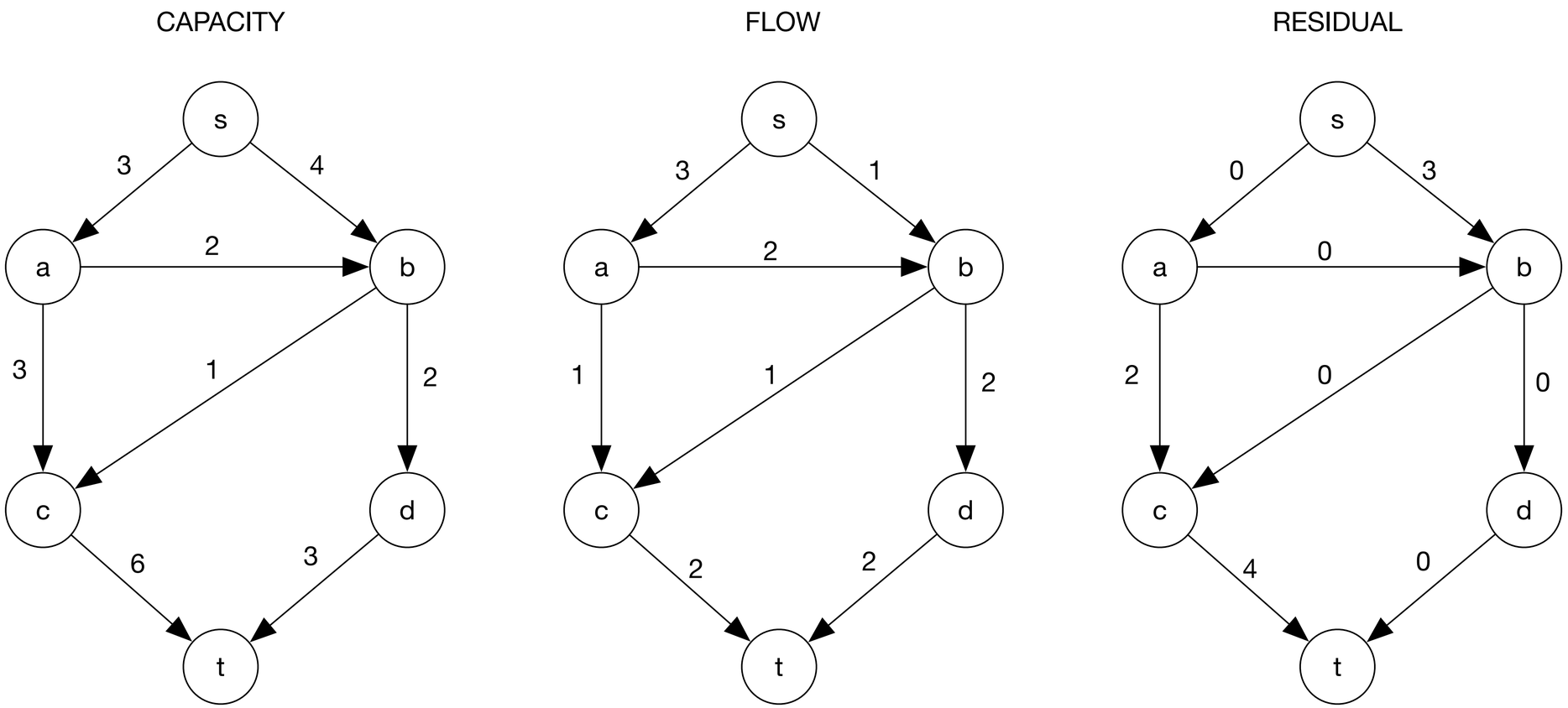
FLOW



RESIDUAL



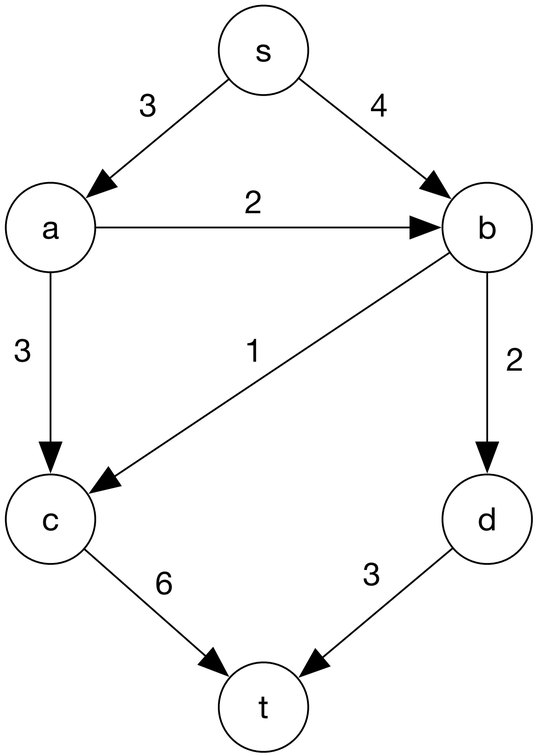
Network flow



Network flow

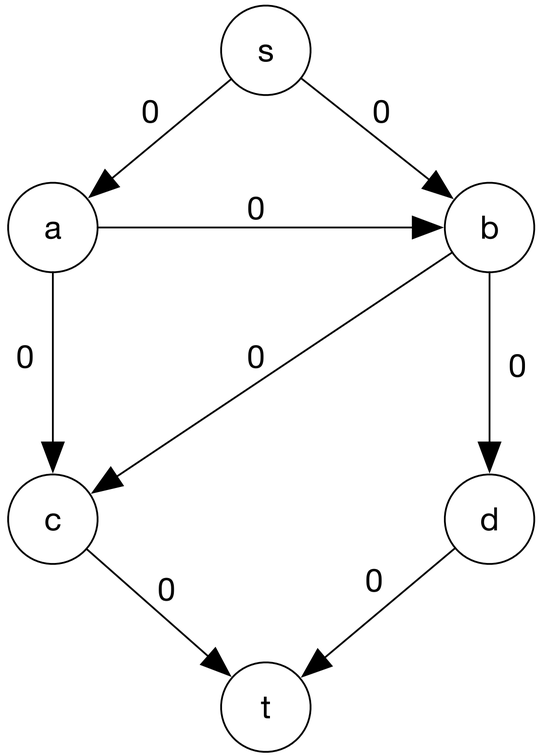
Network flow

CAPACITY



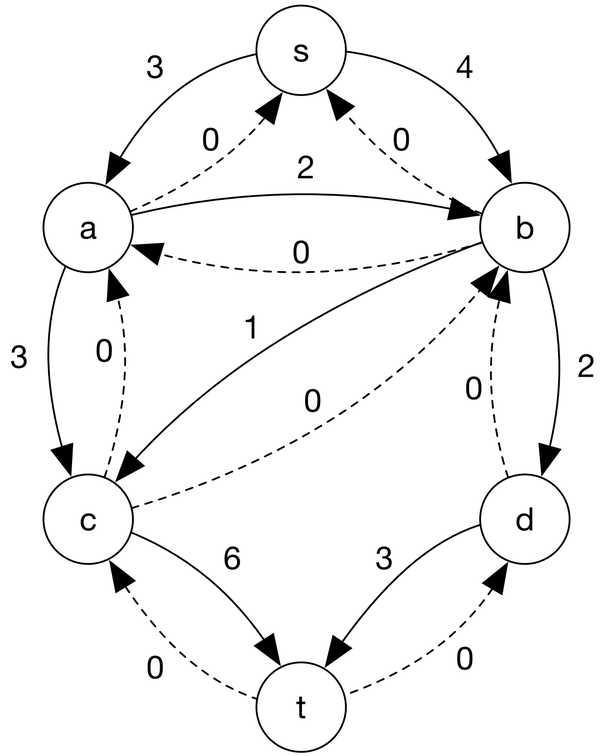
$c_{u,v}$

FLOW



$f_{u,v}$

RESIDUAL

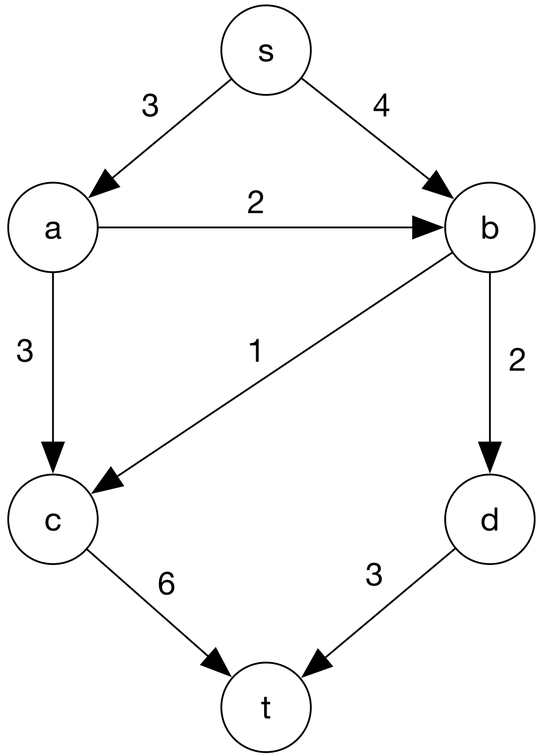


$c_{u,v} - f_{u,v}$

$f_{u,v}$

Network flow

CAPACITY



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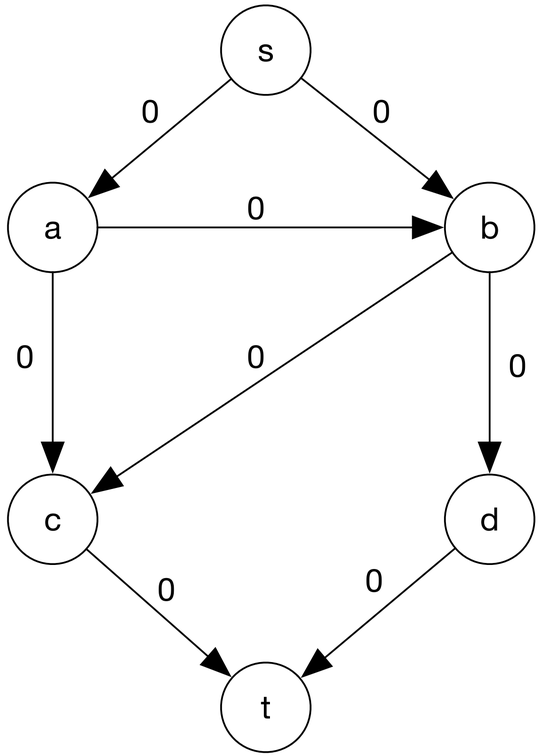

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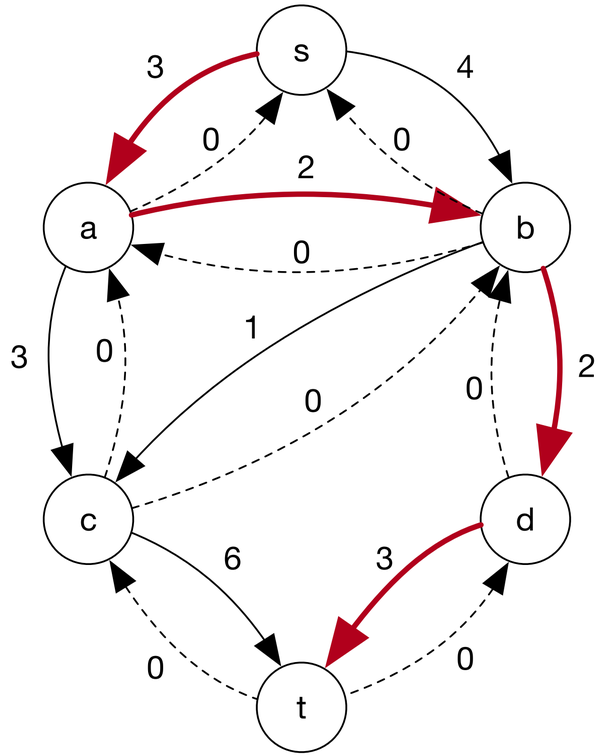
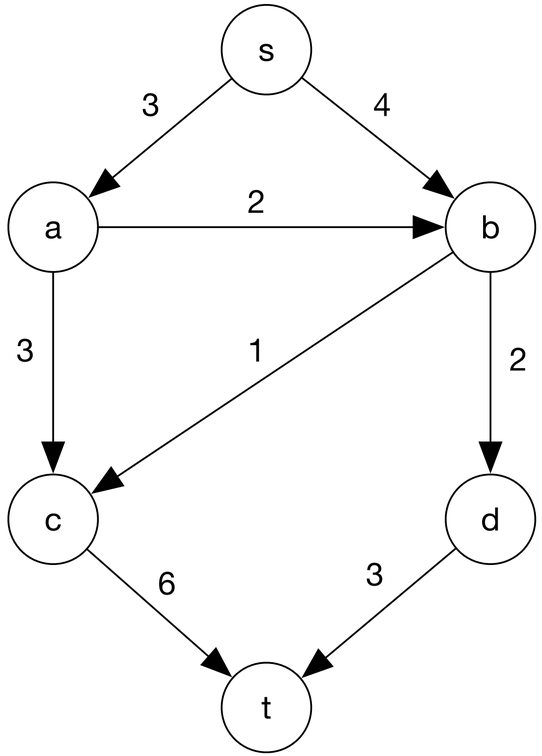


Network flow

CAPACITY



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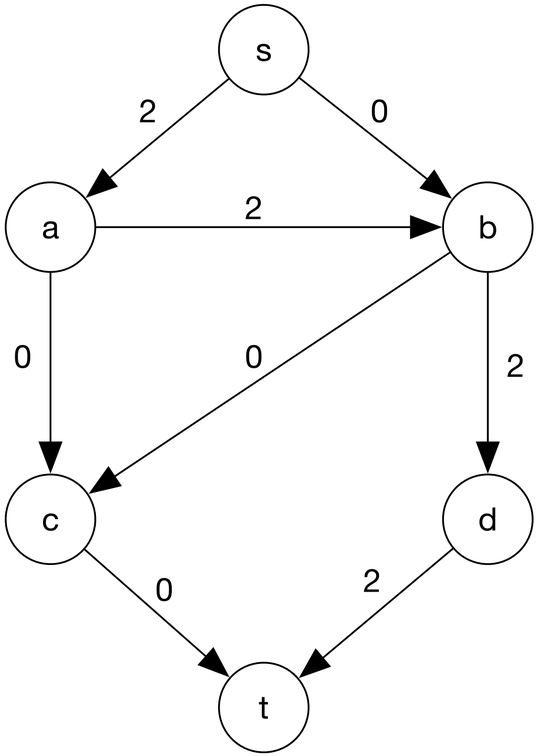

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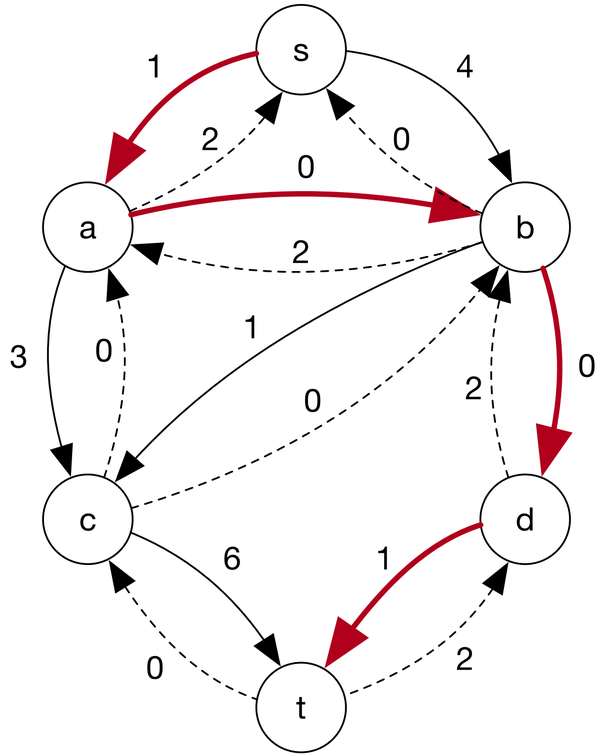
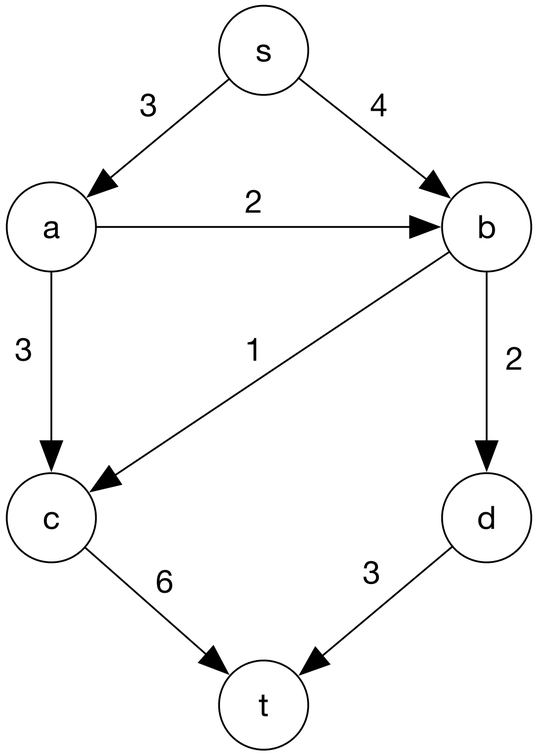


Network flow

CAPACITY



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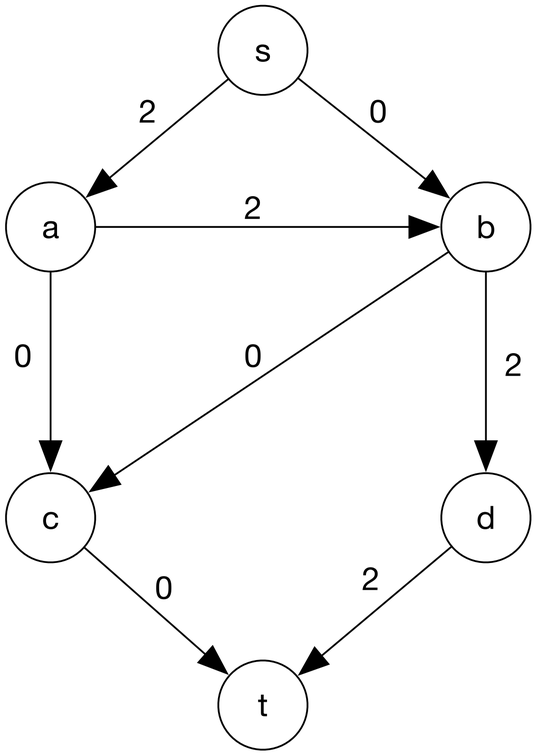

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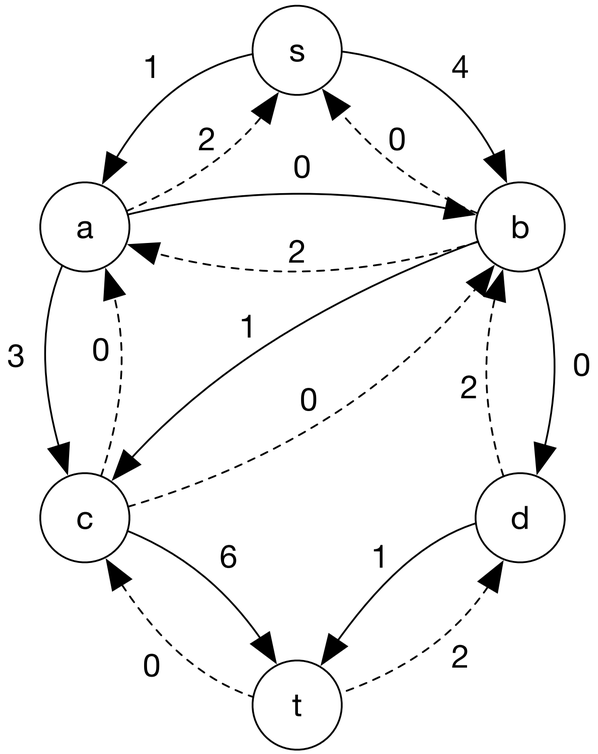
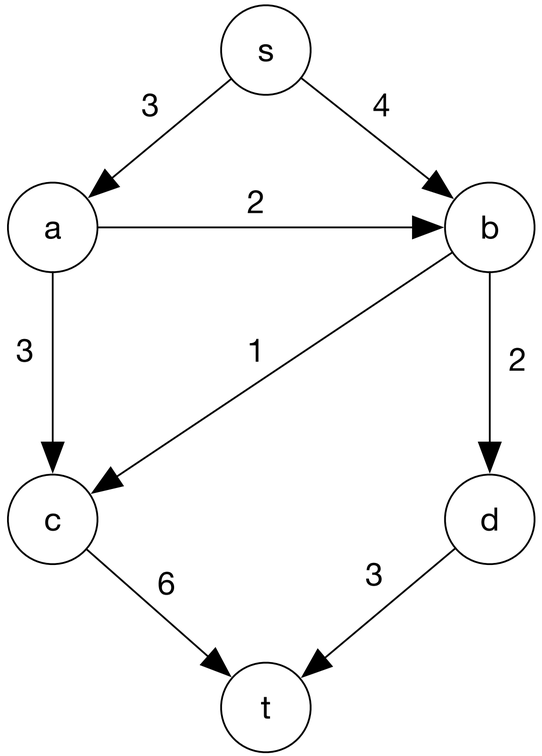


Network flow

CAPACITY



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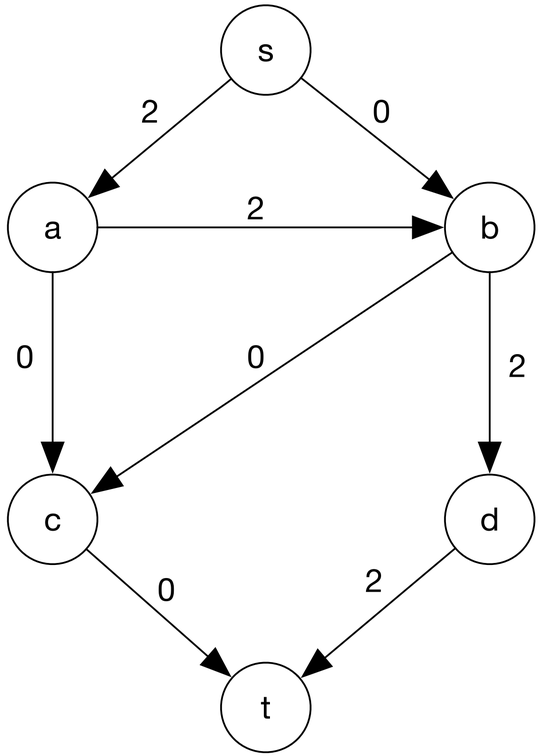

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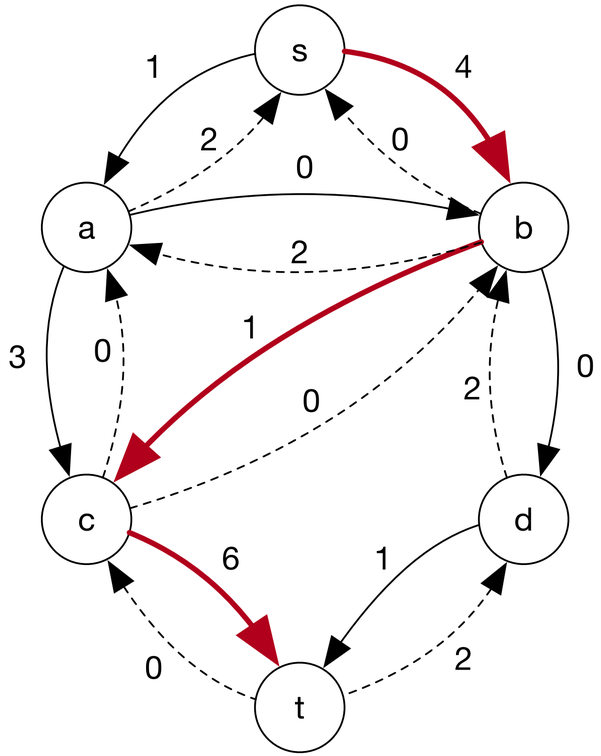
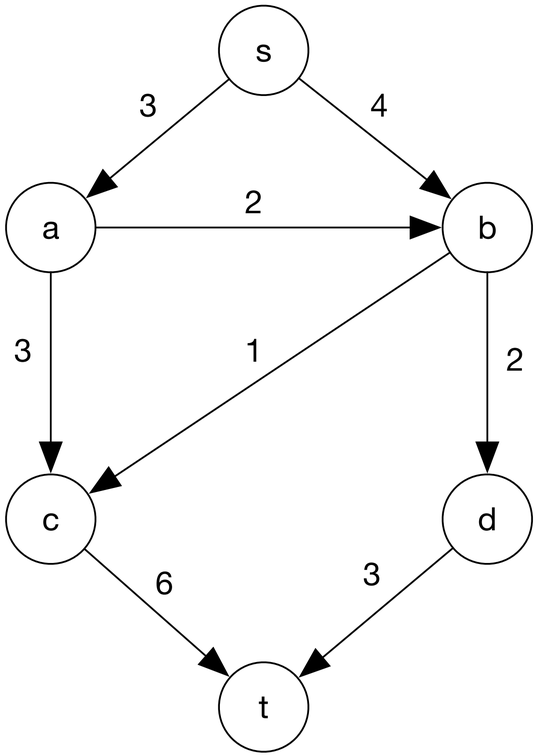


Network flow

CAPACITY



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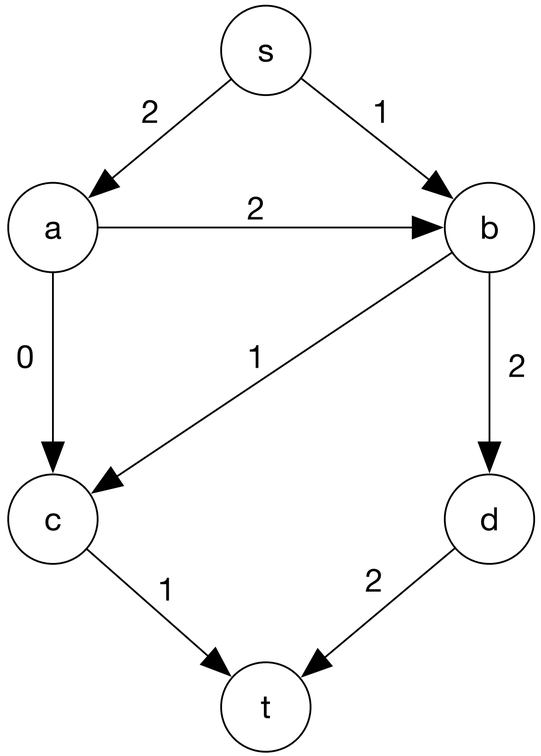

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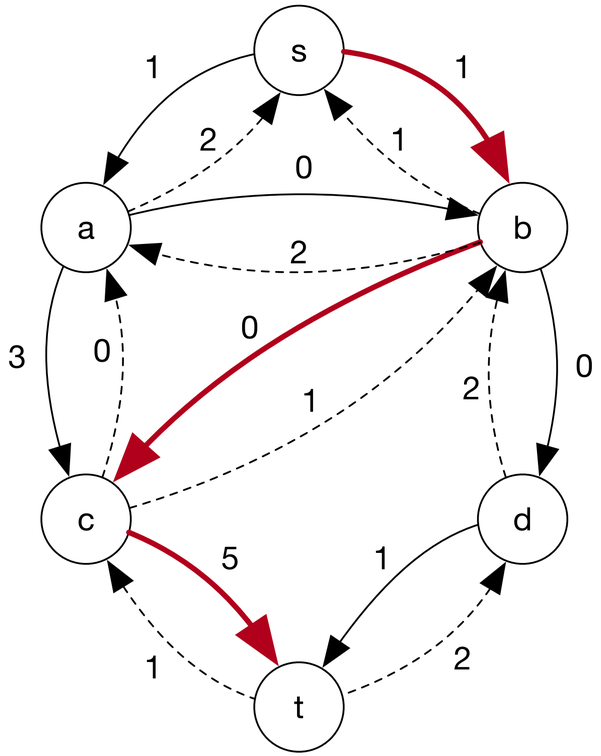
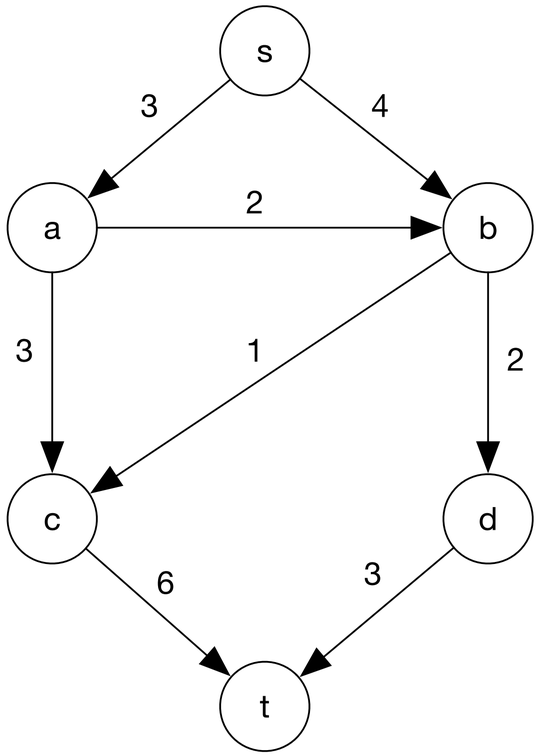


Network flow

CAPACITY



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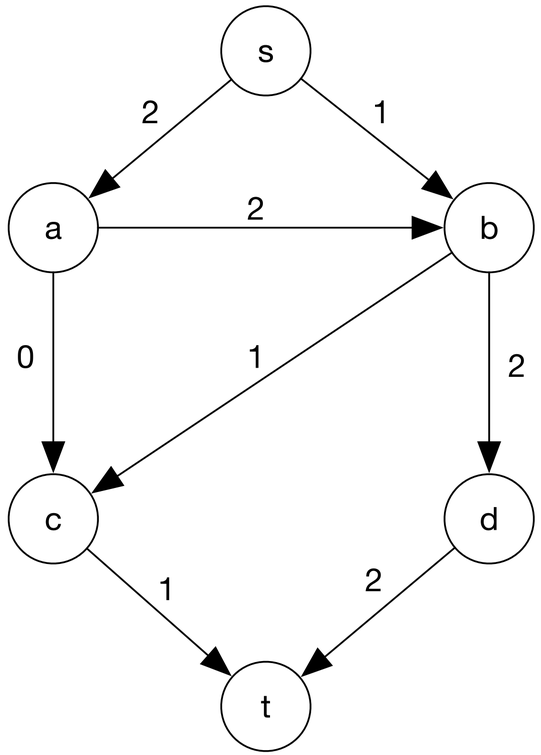

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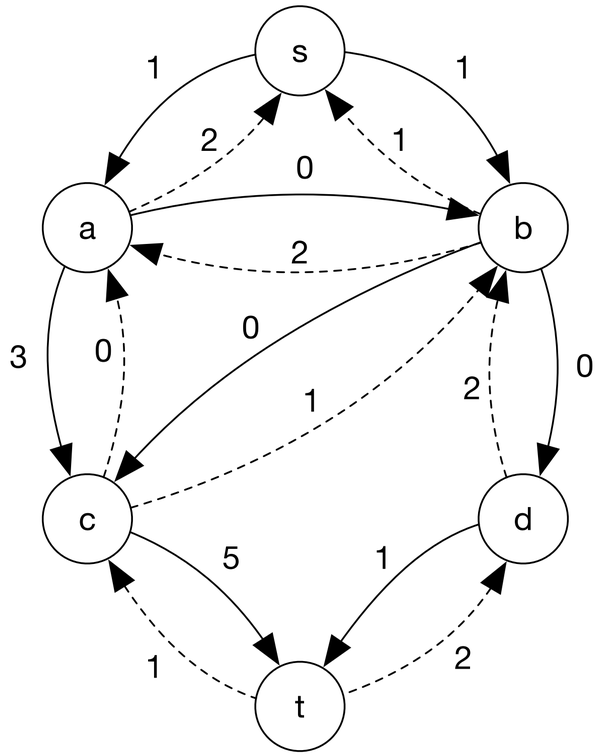
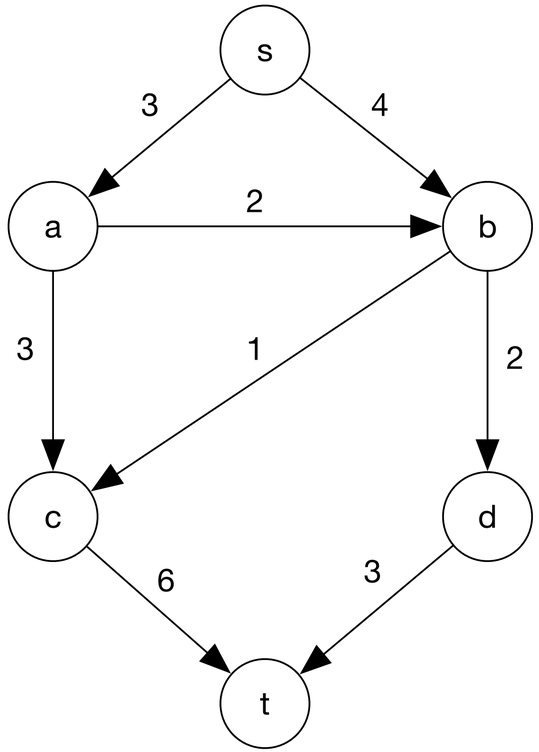


Network flow

CAPACITY



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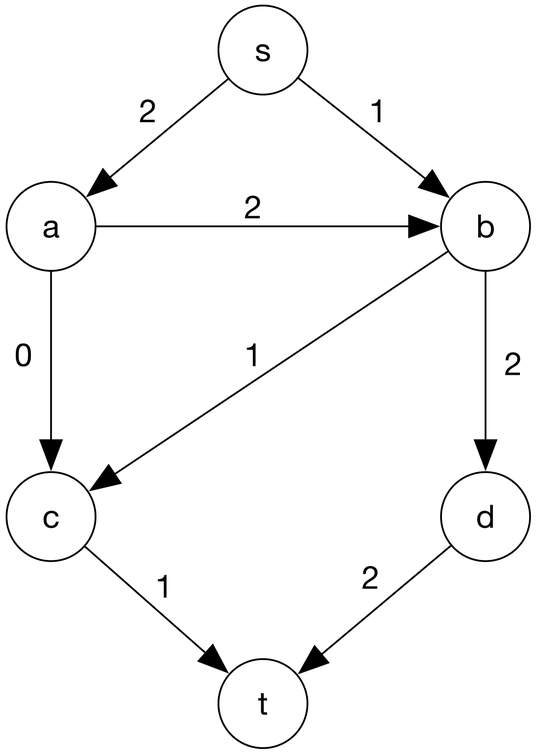

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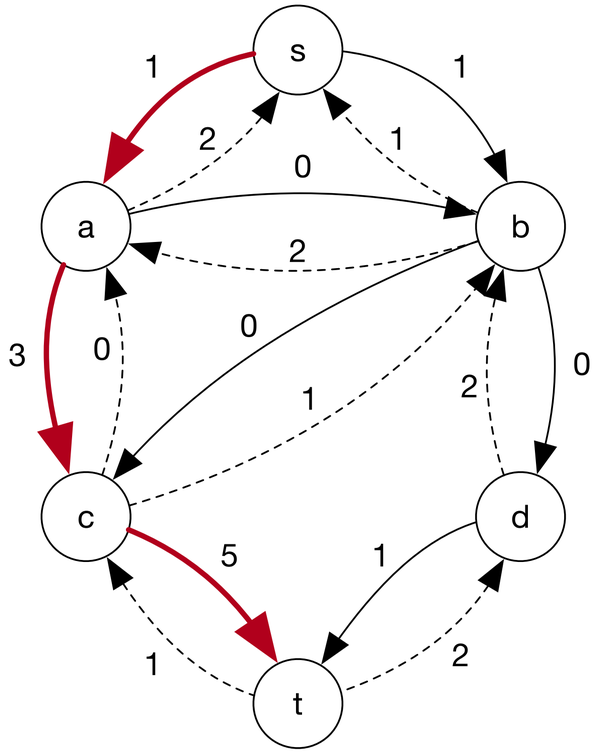
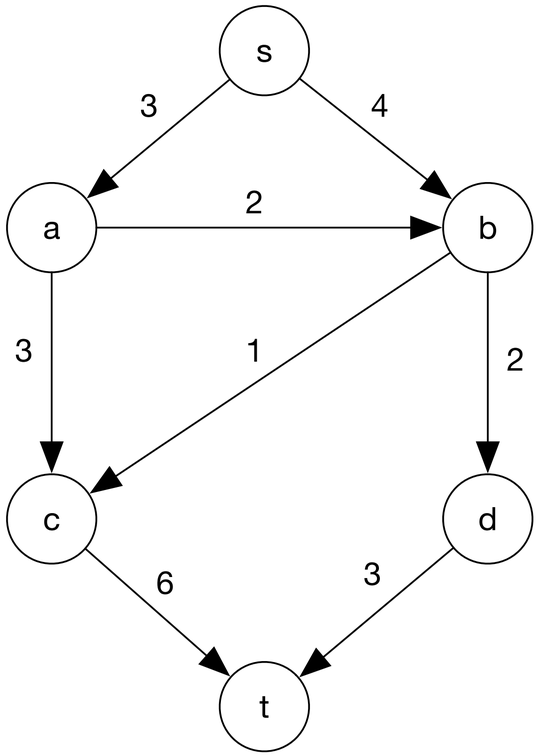


Network flow

CAPACITY



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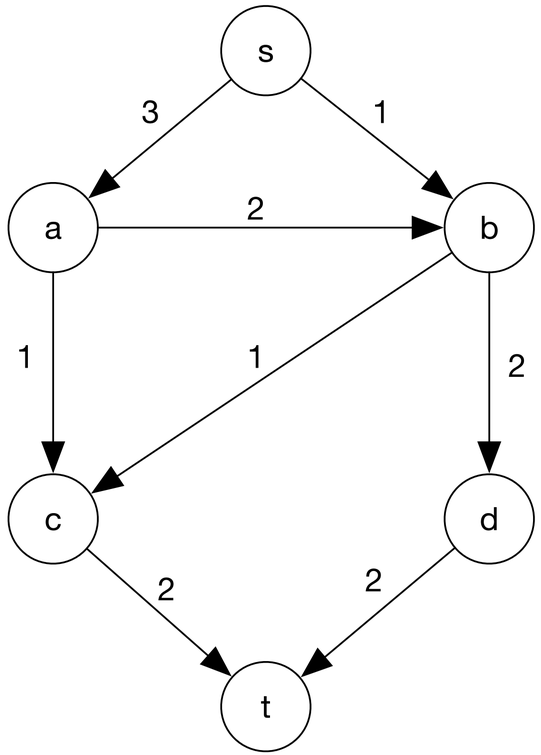

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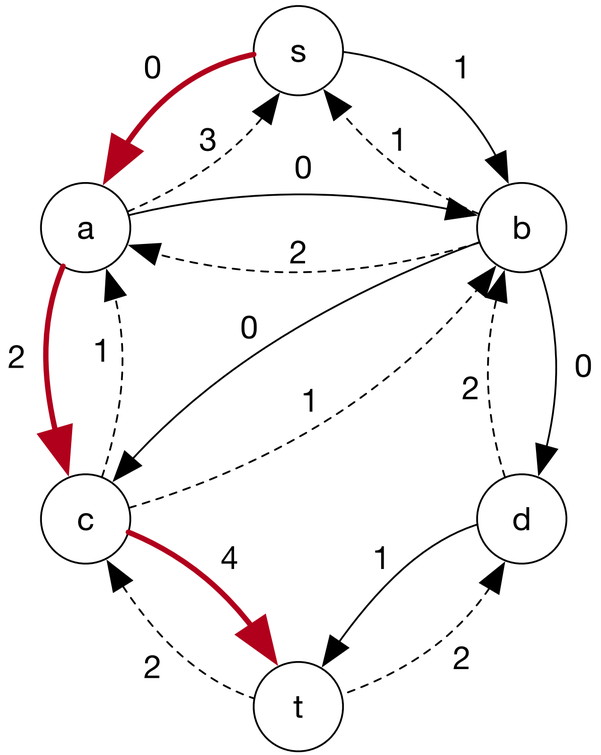
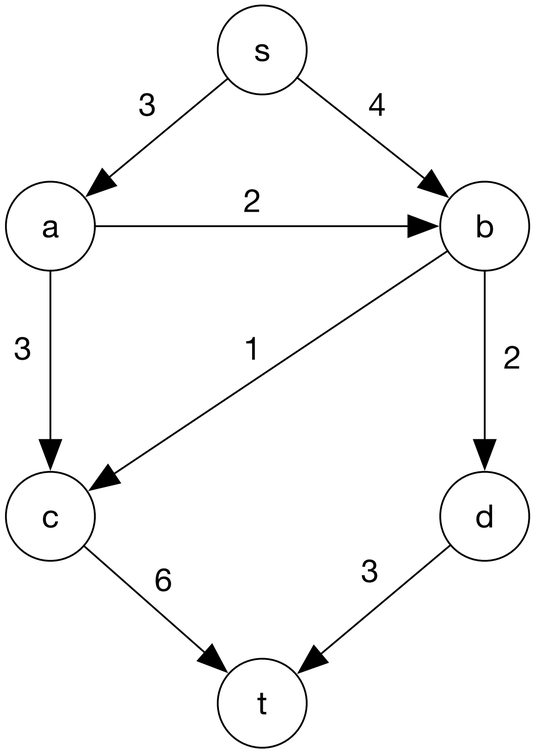


Network flow

CAPACITY



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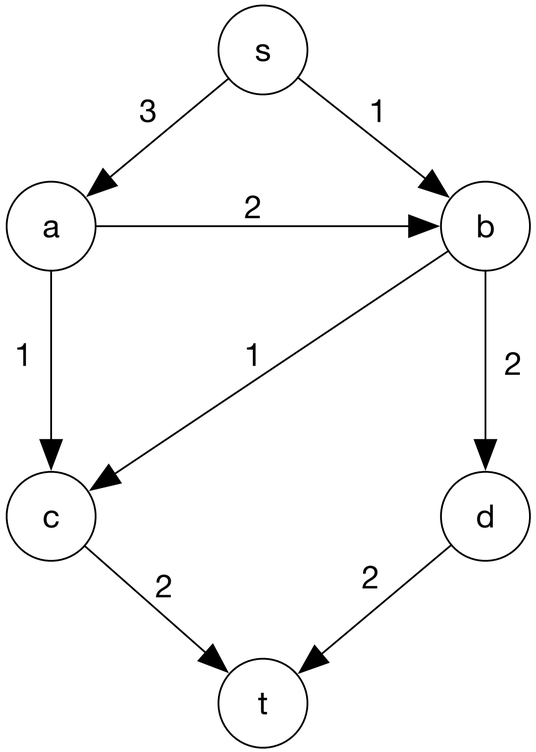

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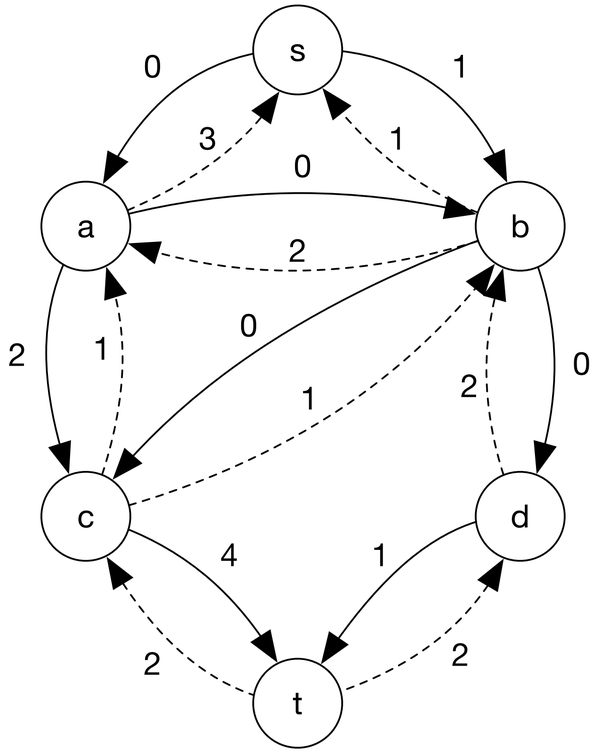
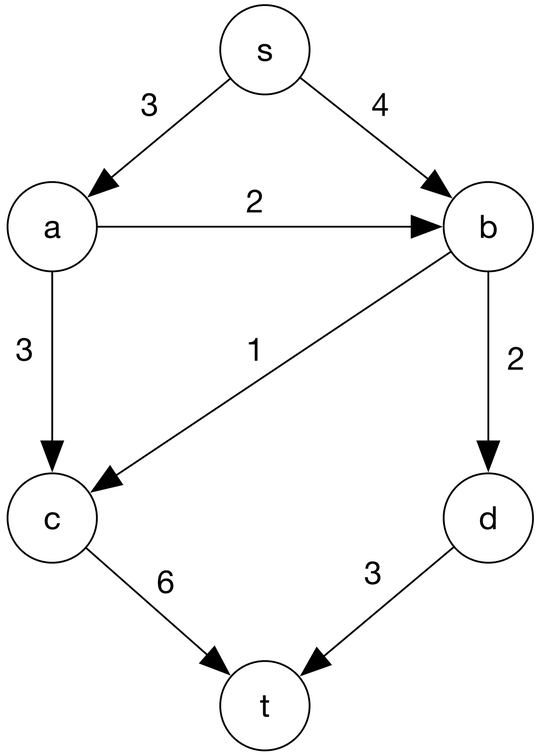


Network flow

CAPACITY



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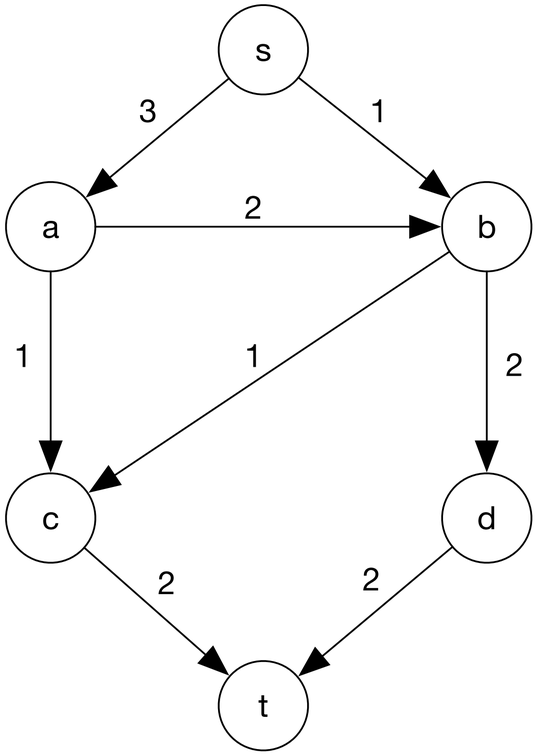

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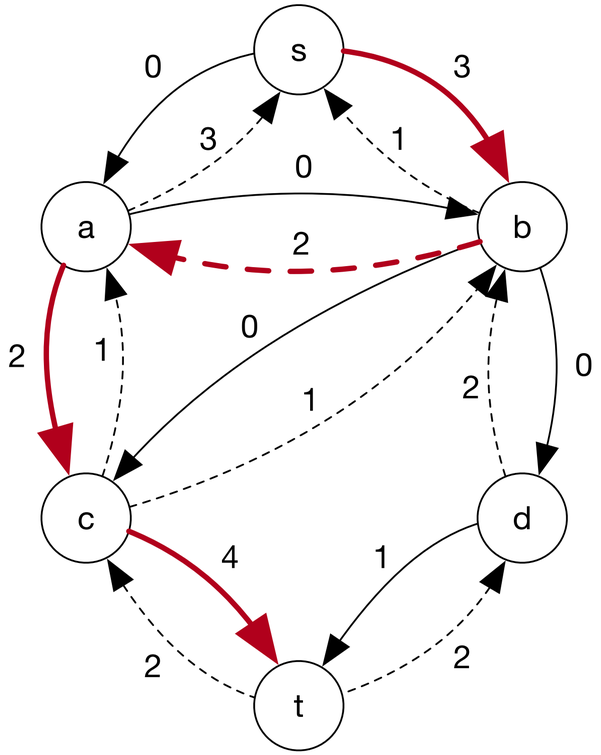
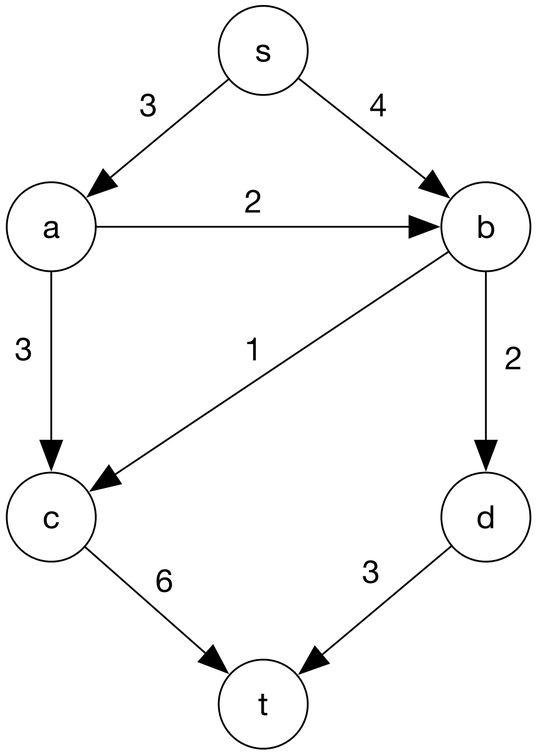


Network flow

CAPACITY



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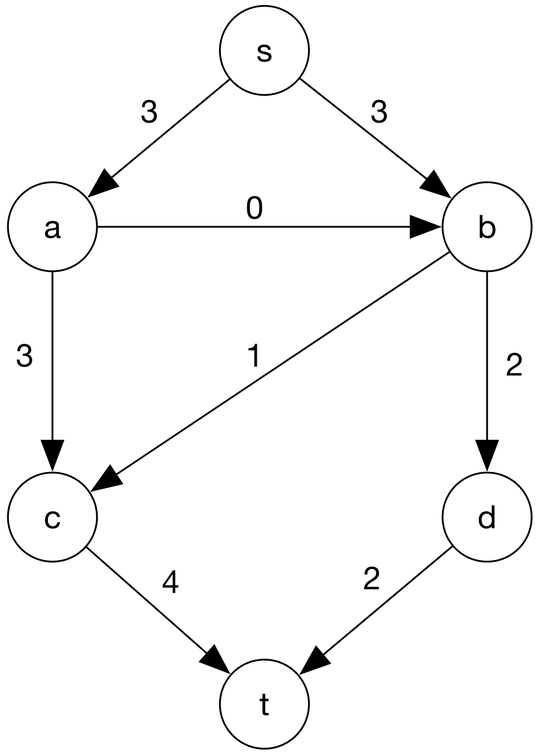

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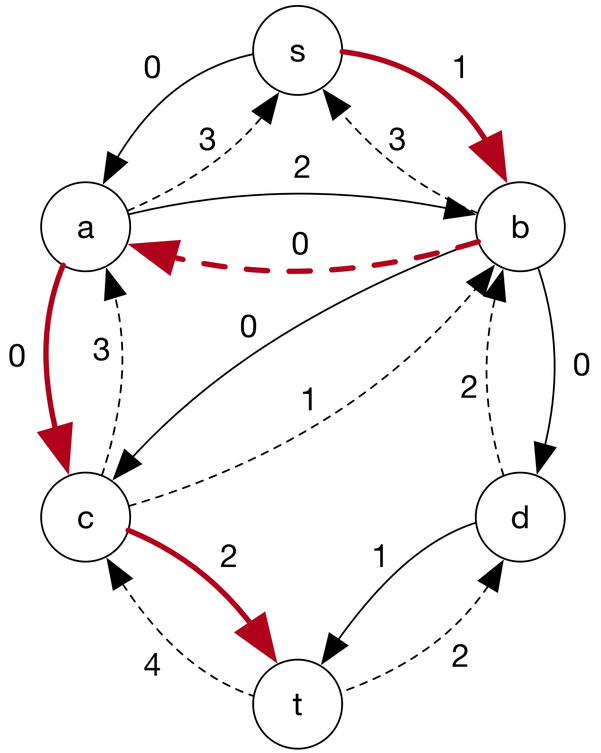
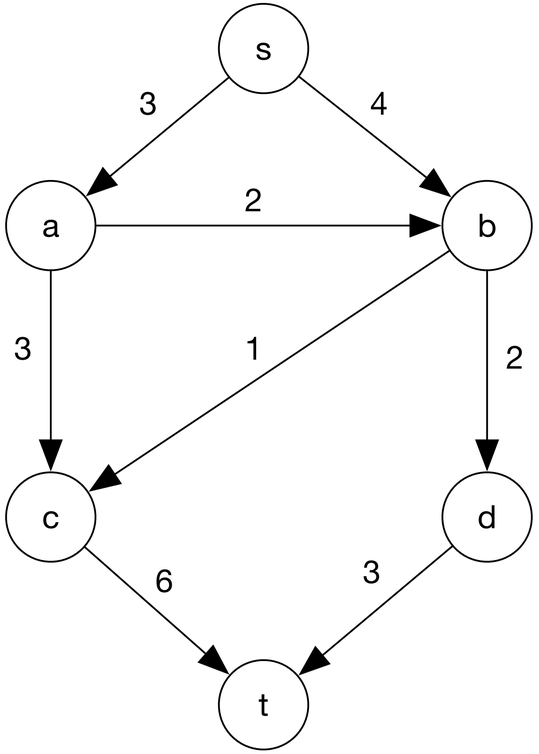
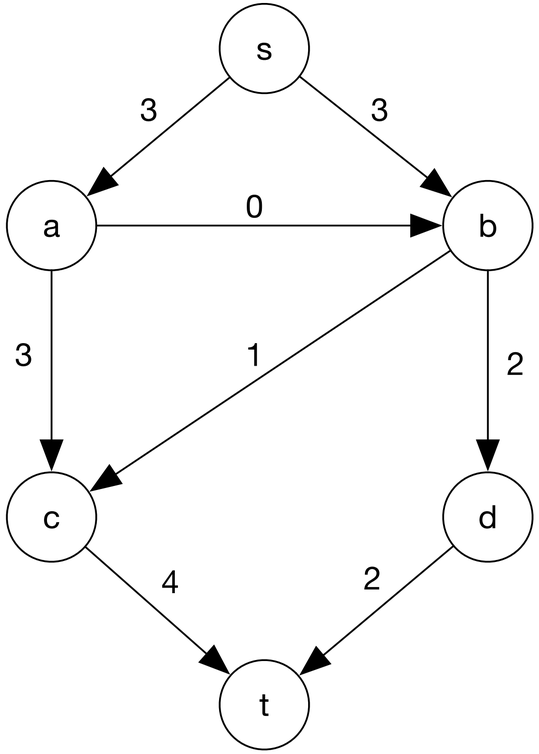


Network flow

CAPACITY



FLOW



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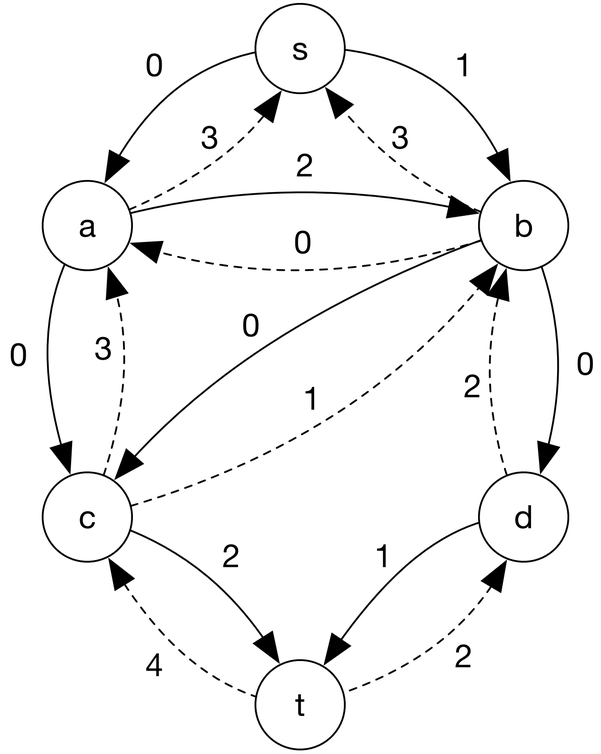

Network flow

Ford-Fulkerson

- Initialize flow network to all zeros
- Initialize residual network with weights on forward edges to correspond capacities, and weights on backward edges to zero
- While augmenting paths exist in the residual graph
  - Update flow network by adding augmenting flow to corresponding edges
  - Update forward edges in residual network by deducting augmenting flow from corresponding edges
  - Update backward edges in residual network by adding augmenting flow to corresponding edges

Network flow

Is this a greedy algorithm? Yes!

What's the complexity? $\mathcal{O}(|E|f)$

How do we find augmenting paths? It depends.

Network flow

Is this a greedy algorithm? Yes!

What's the complexity? $\mathcal{O}(|E|f)$

How do we find augmenting paths? It depends.

- If we use breadth-first search, this is called Edmonds-Karp algorithm
- Worst case complexity of Edmonds-Karp $\mathcal{O}(|V| \times |E|^2)$

Network flow

What we haven't discussed:

- Reals: rational- and irrational-valued capacities
- Proof of correctness
- Special applications of network flow (*i.e.*, bipartite matching)