



THE UNIVERSITY OF VERMONT
COLLEGE OF ENGINEERING &
MATHEMATICAL SCIENCES

Minimum Spanning Tree

Prim's algorithm

Minimum spanning tree

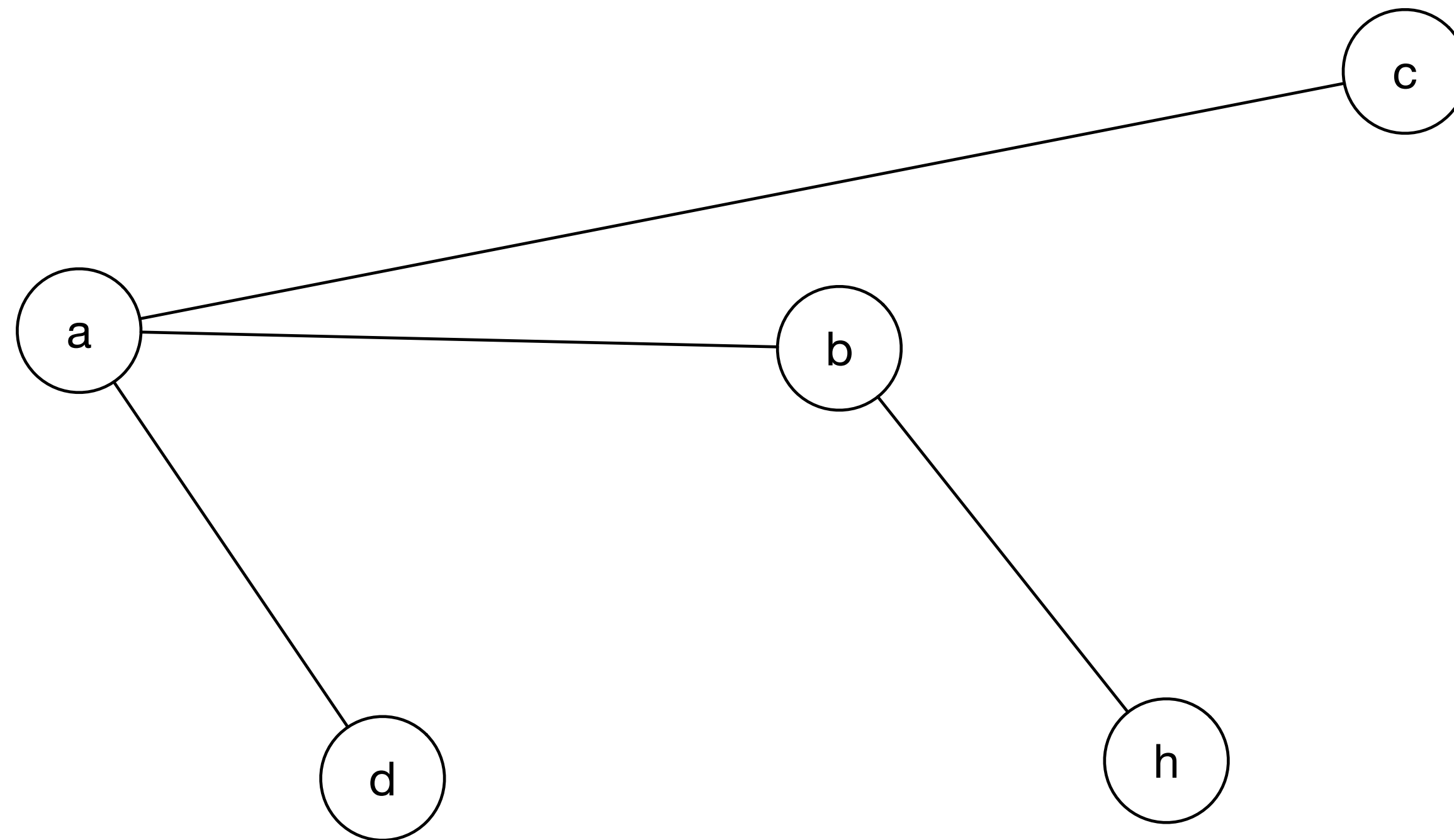
Trees: review

What is a tree?

Minimum spanning tree

Trees: review

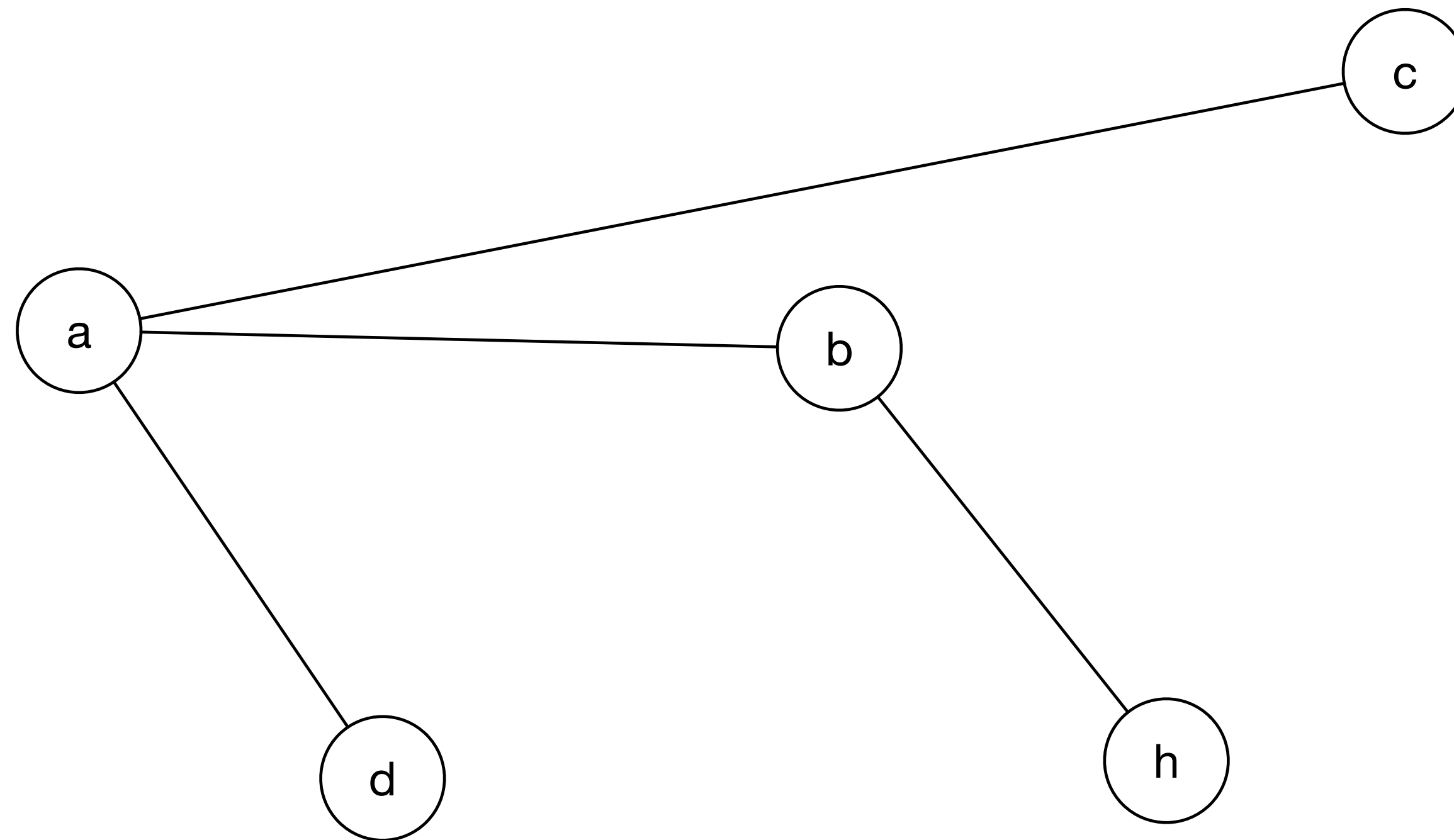
What is a tree? An acyclic, connected, undirected graph



Minimum spanning tree

Trees: review

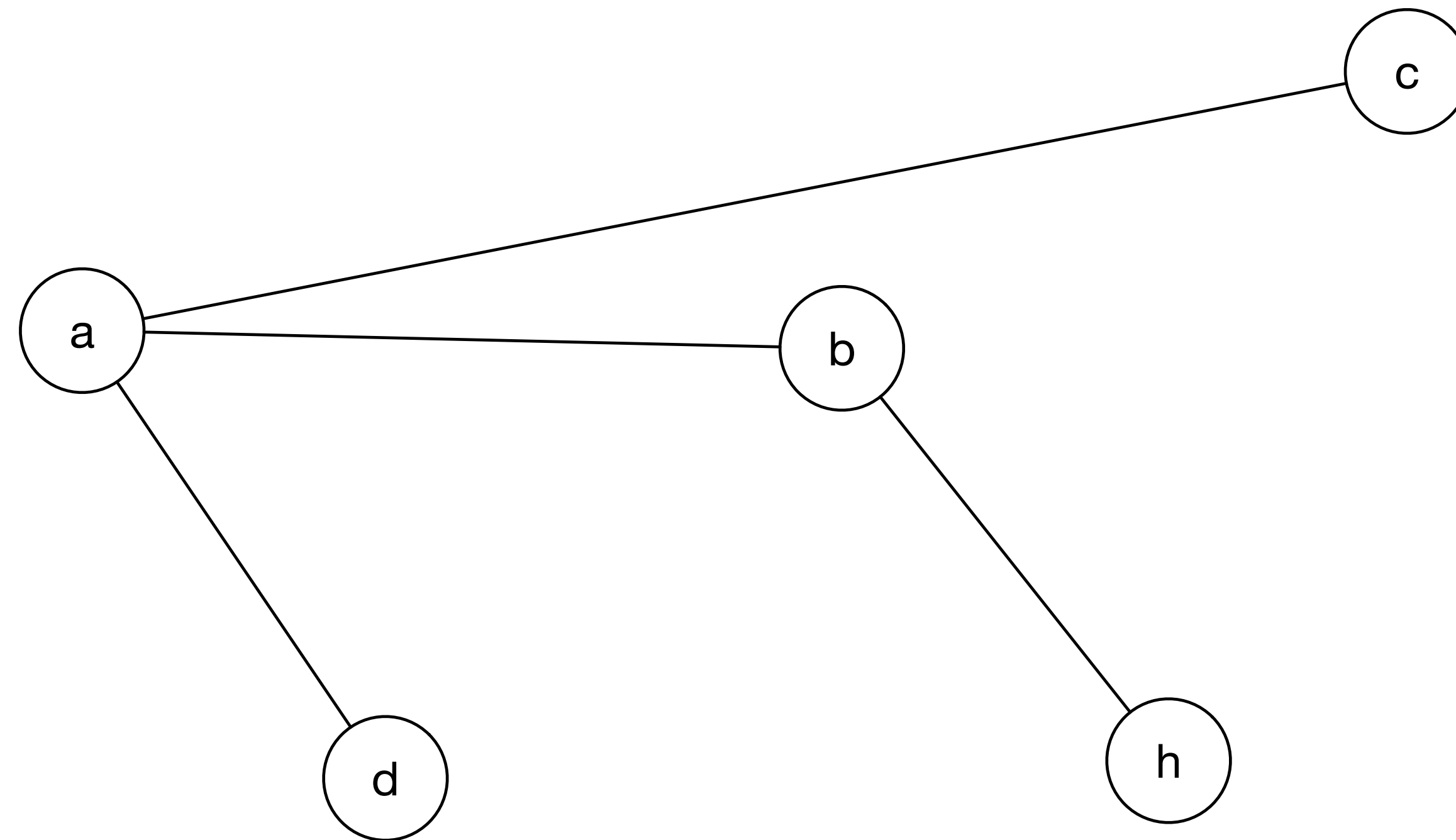
What is a tree? An acyclic, connected, undirected graph



Minimum spanning tree

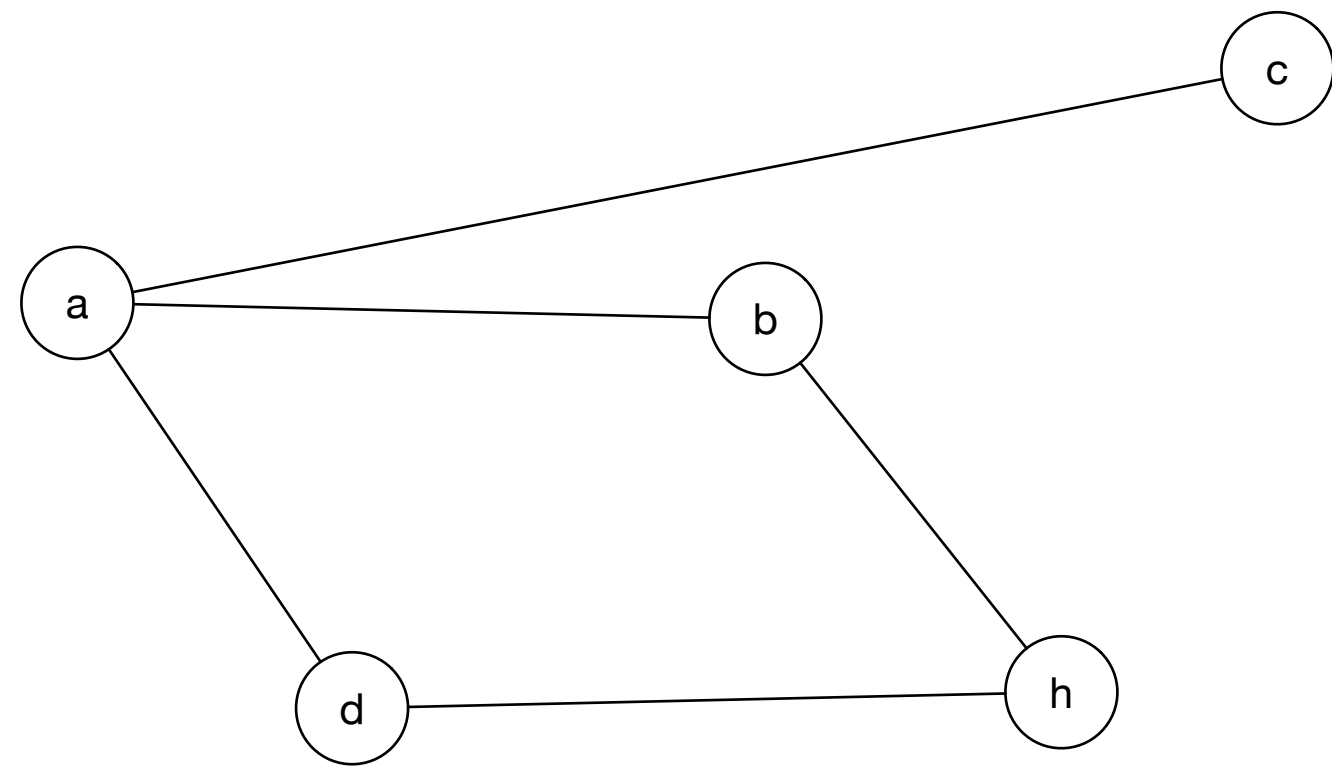
Trees: review

What is a tree? An acyclic, connected, undirected graph

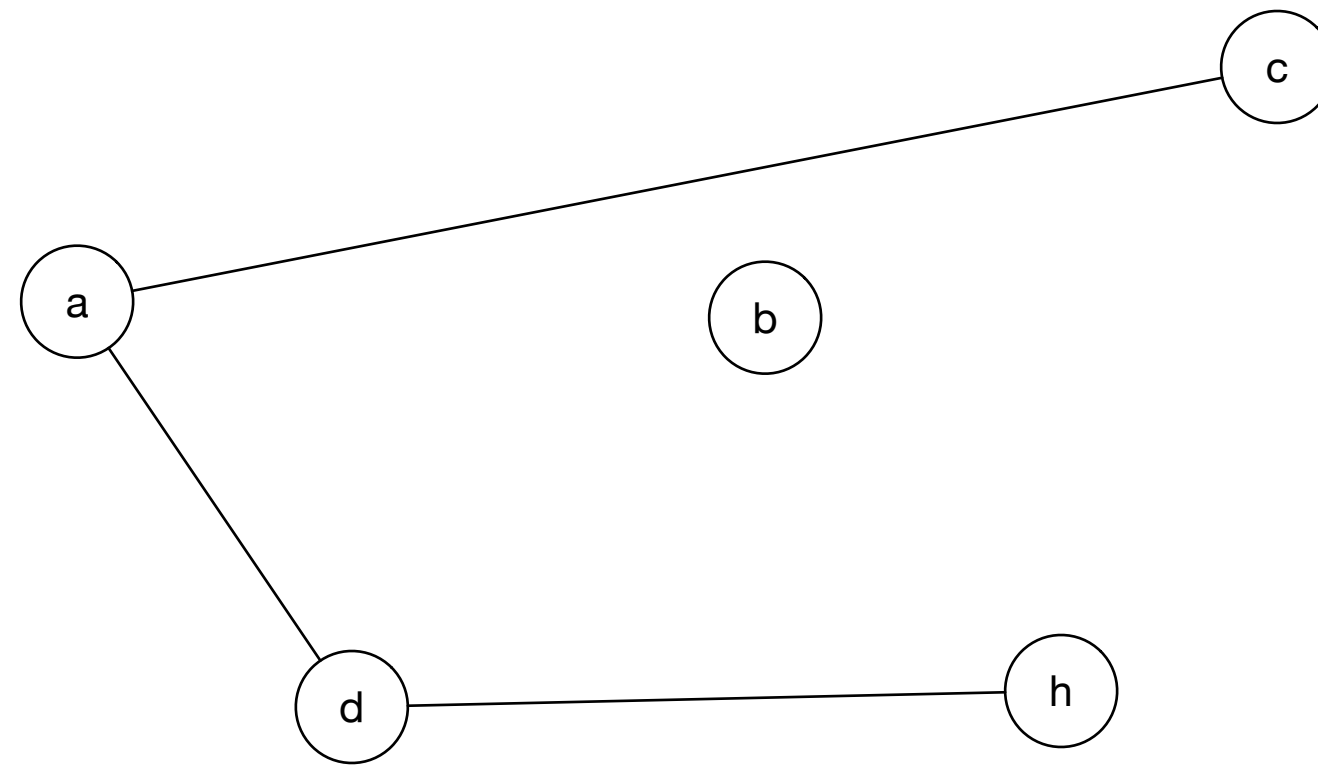


Minimum spanning tree

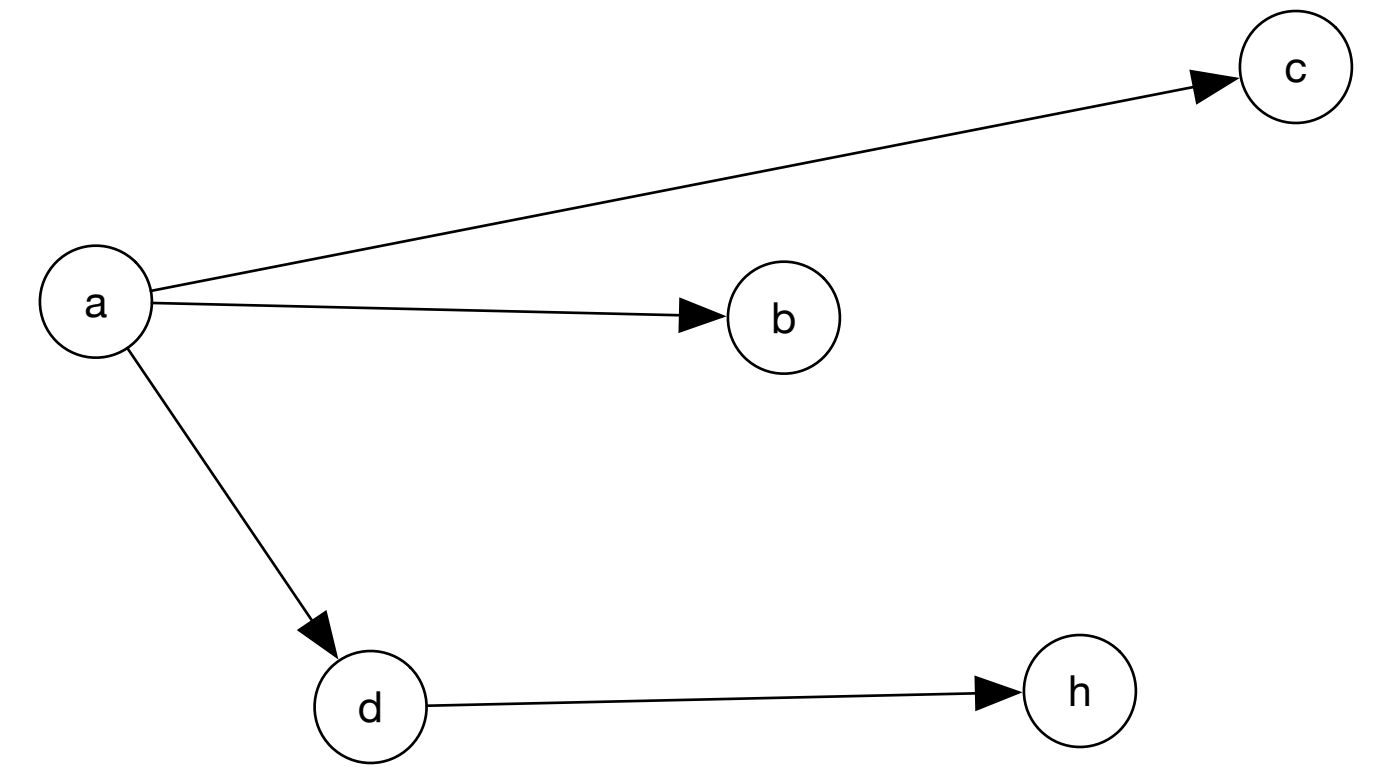
Trees: review



not acyclic



not connected

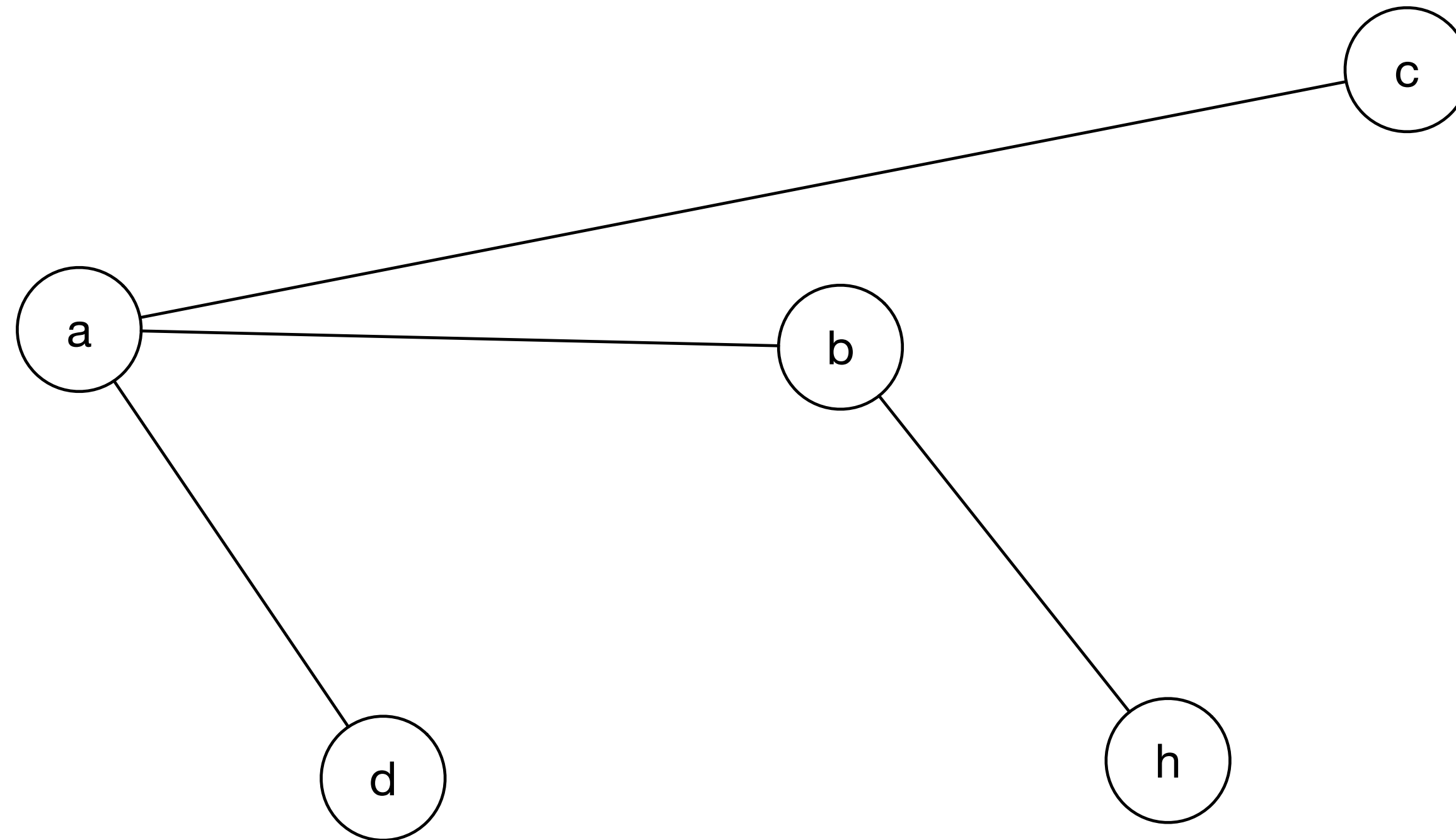


not undirected

Minimum spanning tree

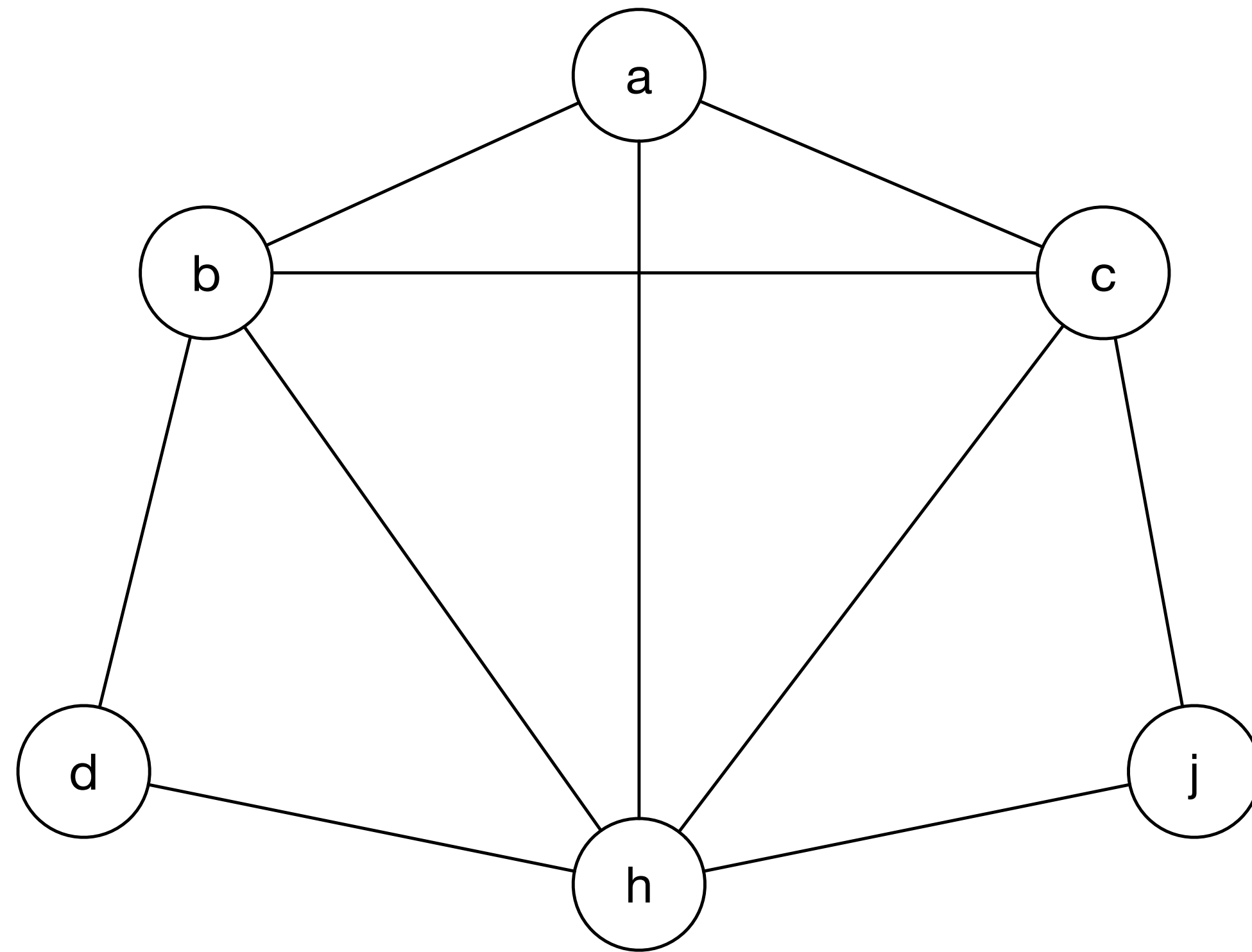
Trees: review

On n nodes, a tree has $n - 1$ edges



Minimum spanning tree

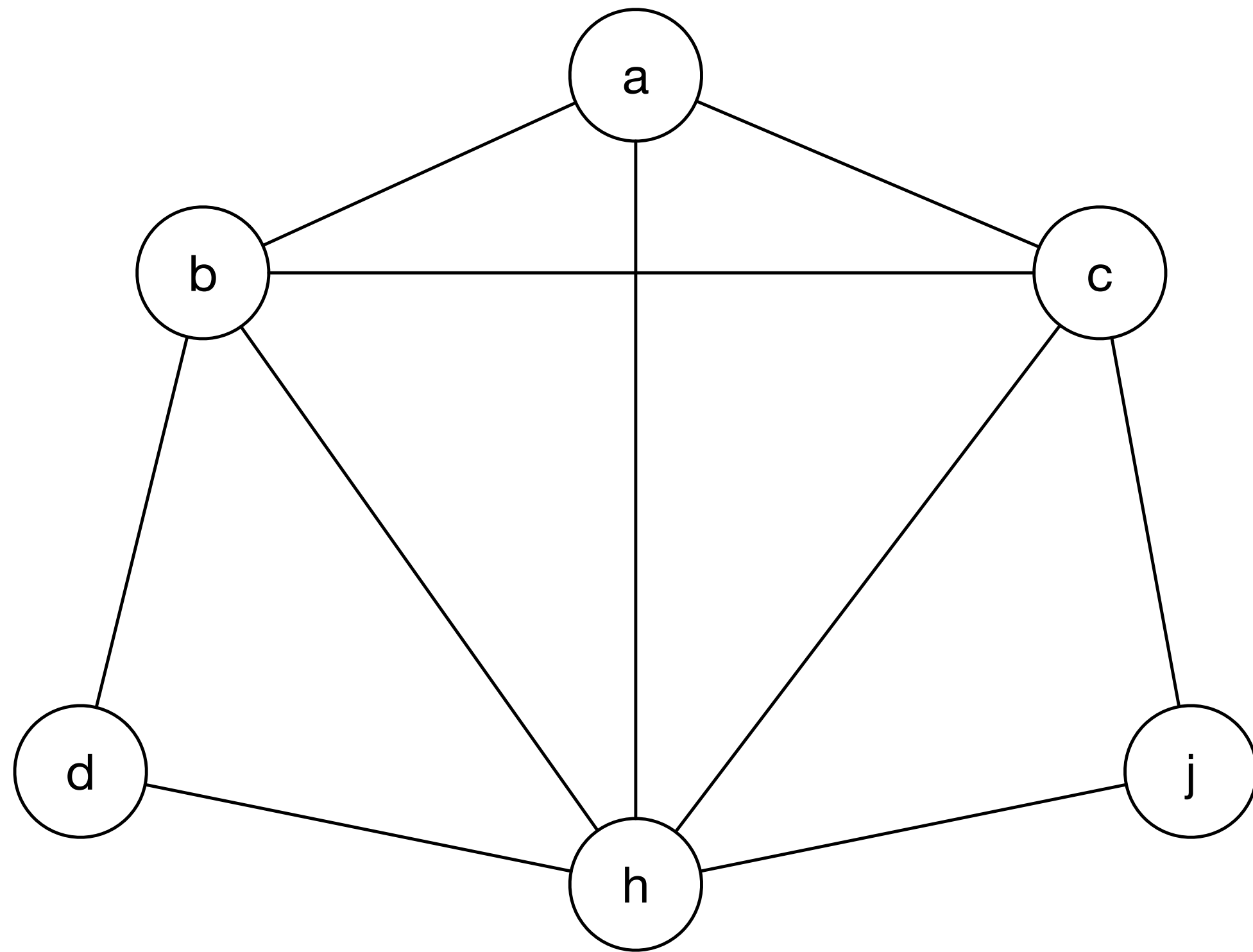
Spanning tree



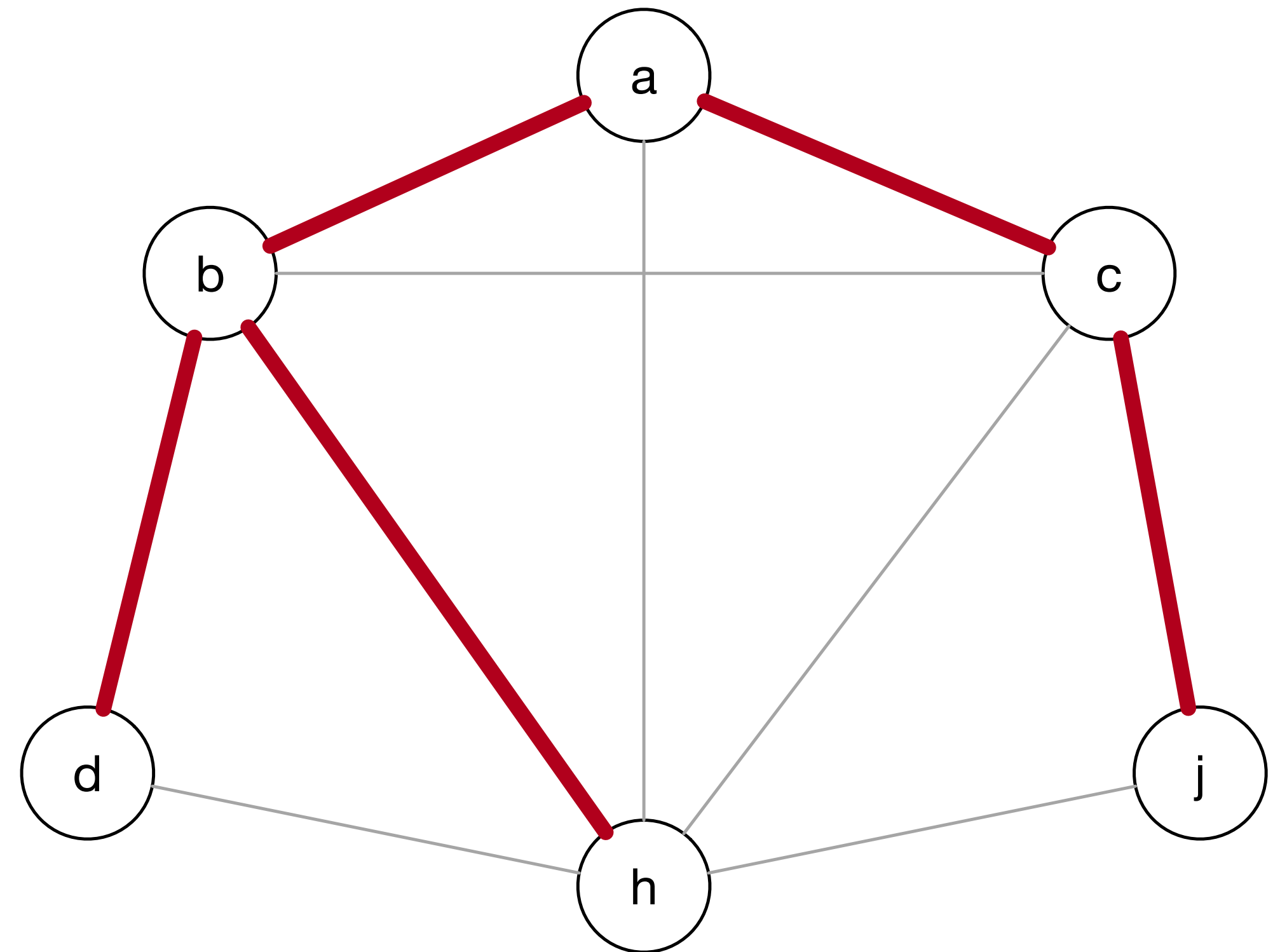
$$G = (V, E)$$

Minimum spanning tree

Spanning tree



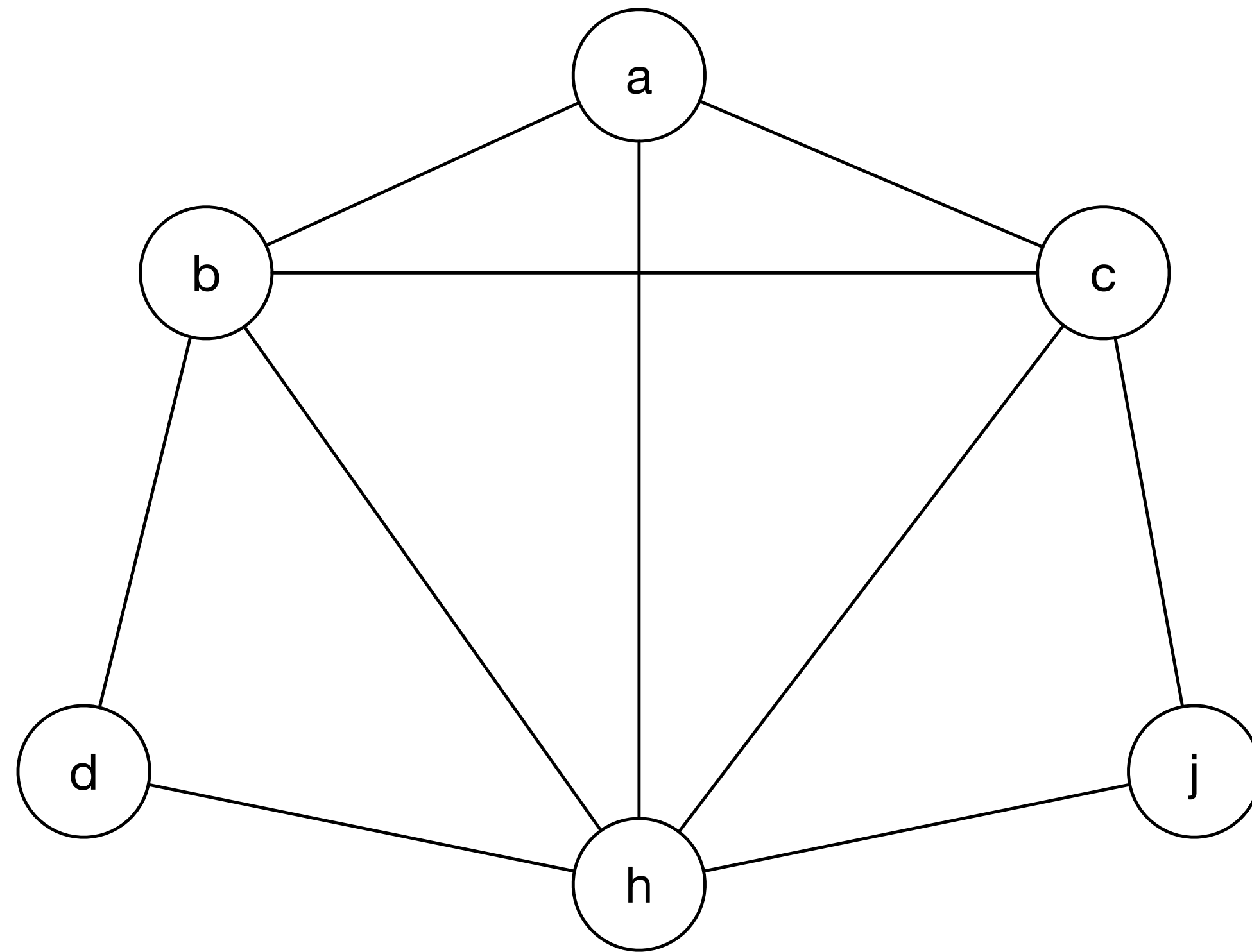
$$G = (V, E)$$



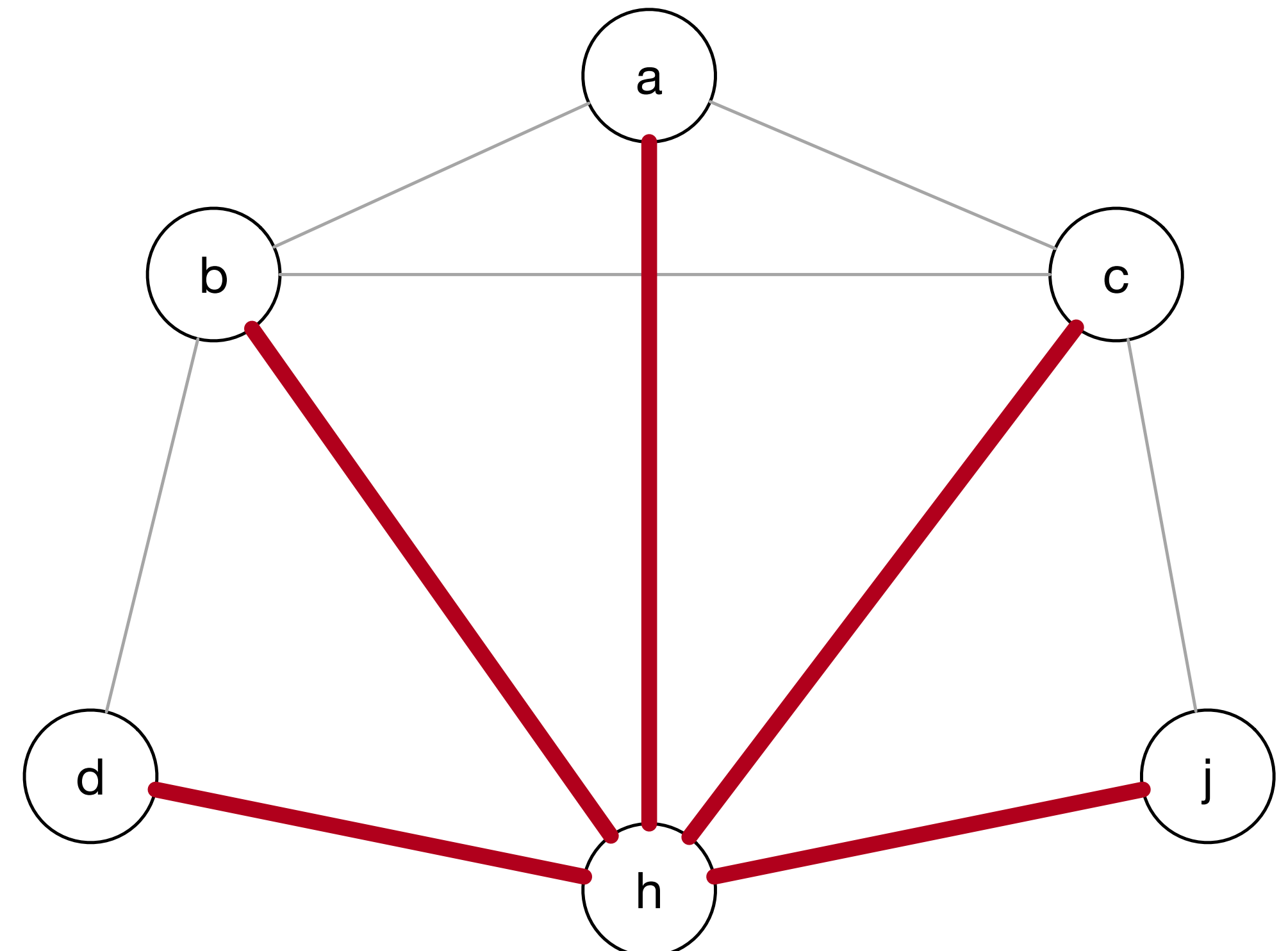
$$G' = (V, E') \text{ with } E' \subseteq E$$

Minimum spanning tree

Spanning tree

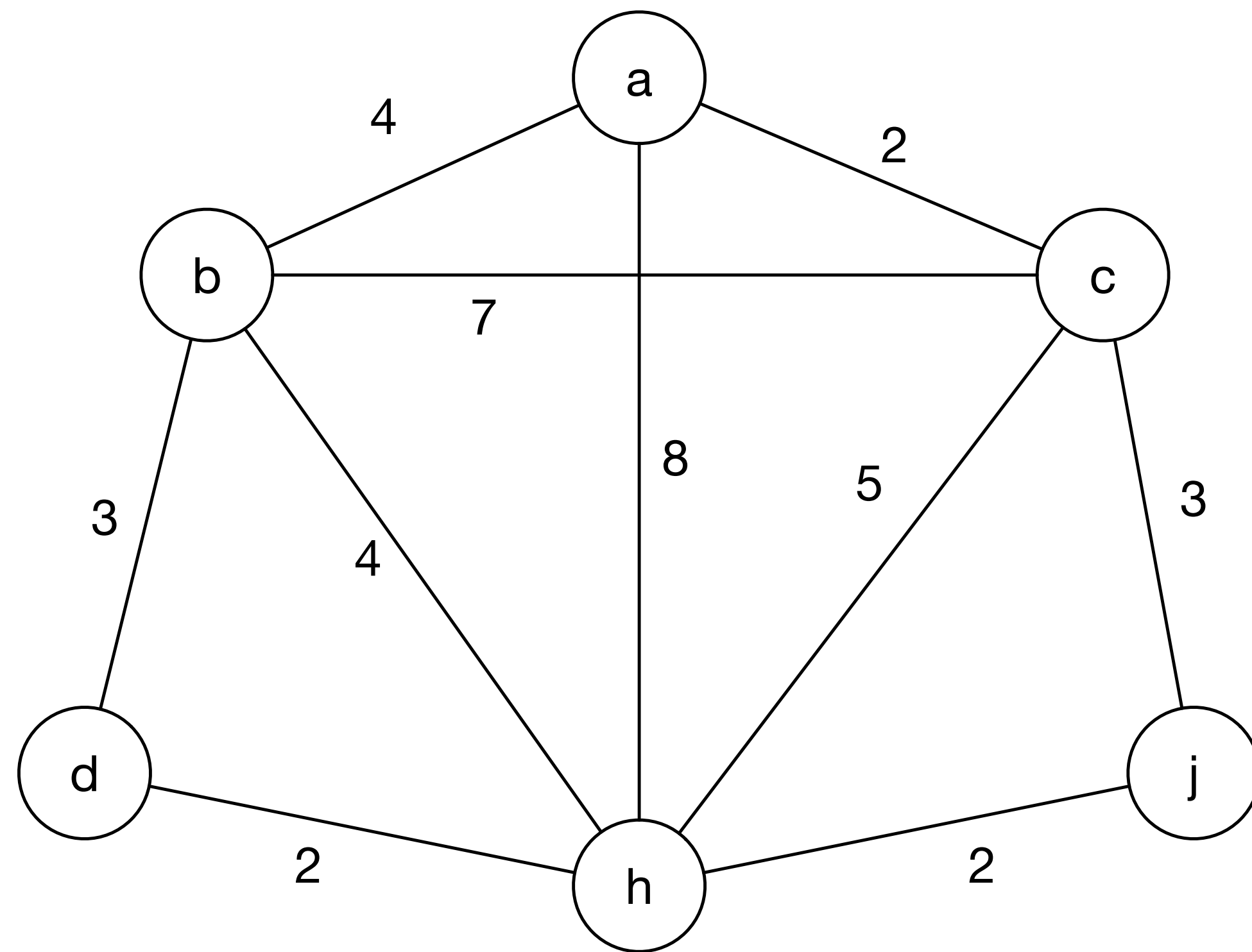


$$G = (V, E)$$



$$G' = (V, E') \text{ with } E' \subseteq E$$

Minimum spanning tree



$$G = (V, E)$$

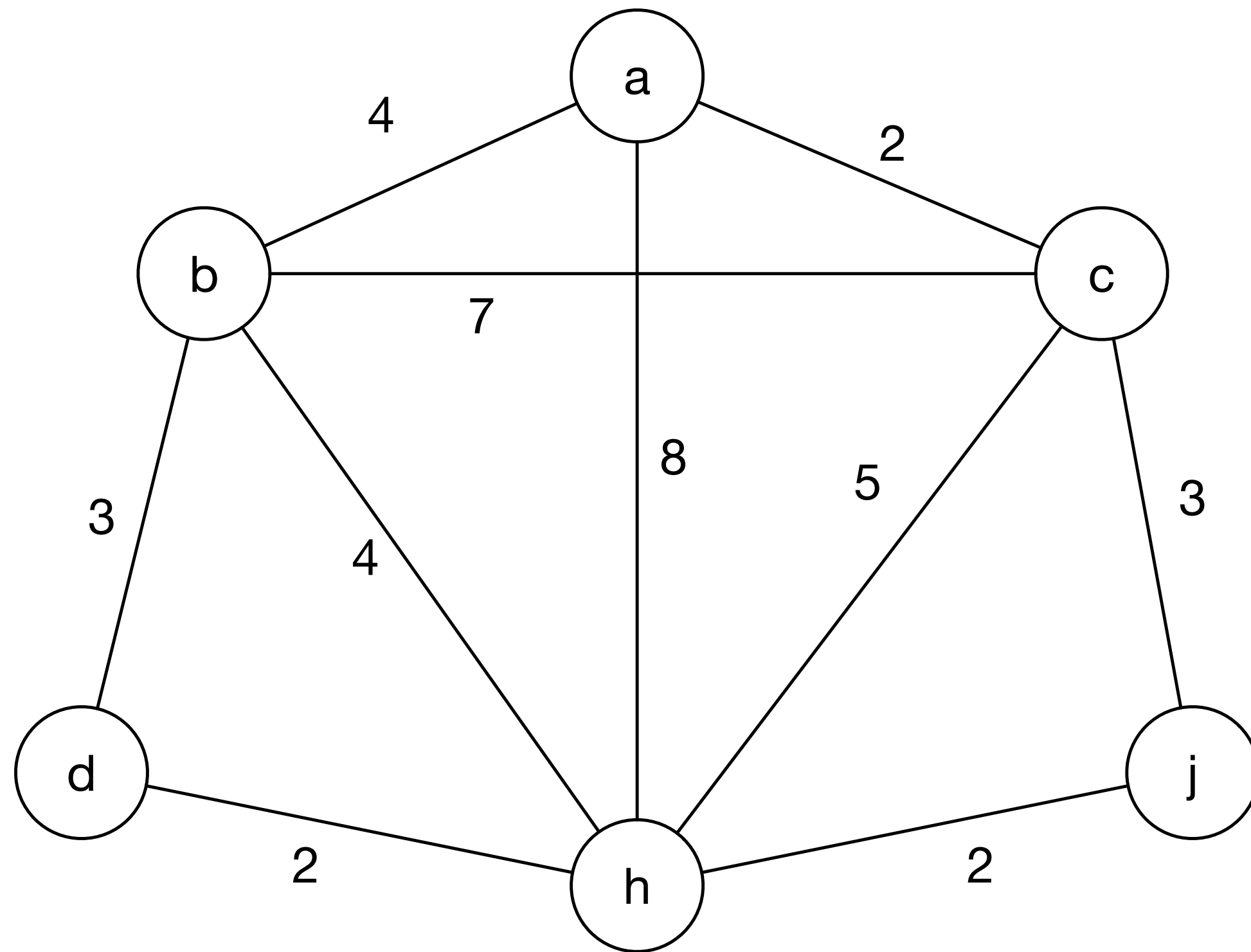
$$G' = (V, E') \text{ with } E' \subseteq E$$

G' is a tree

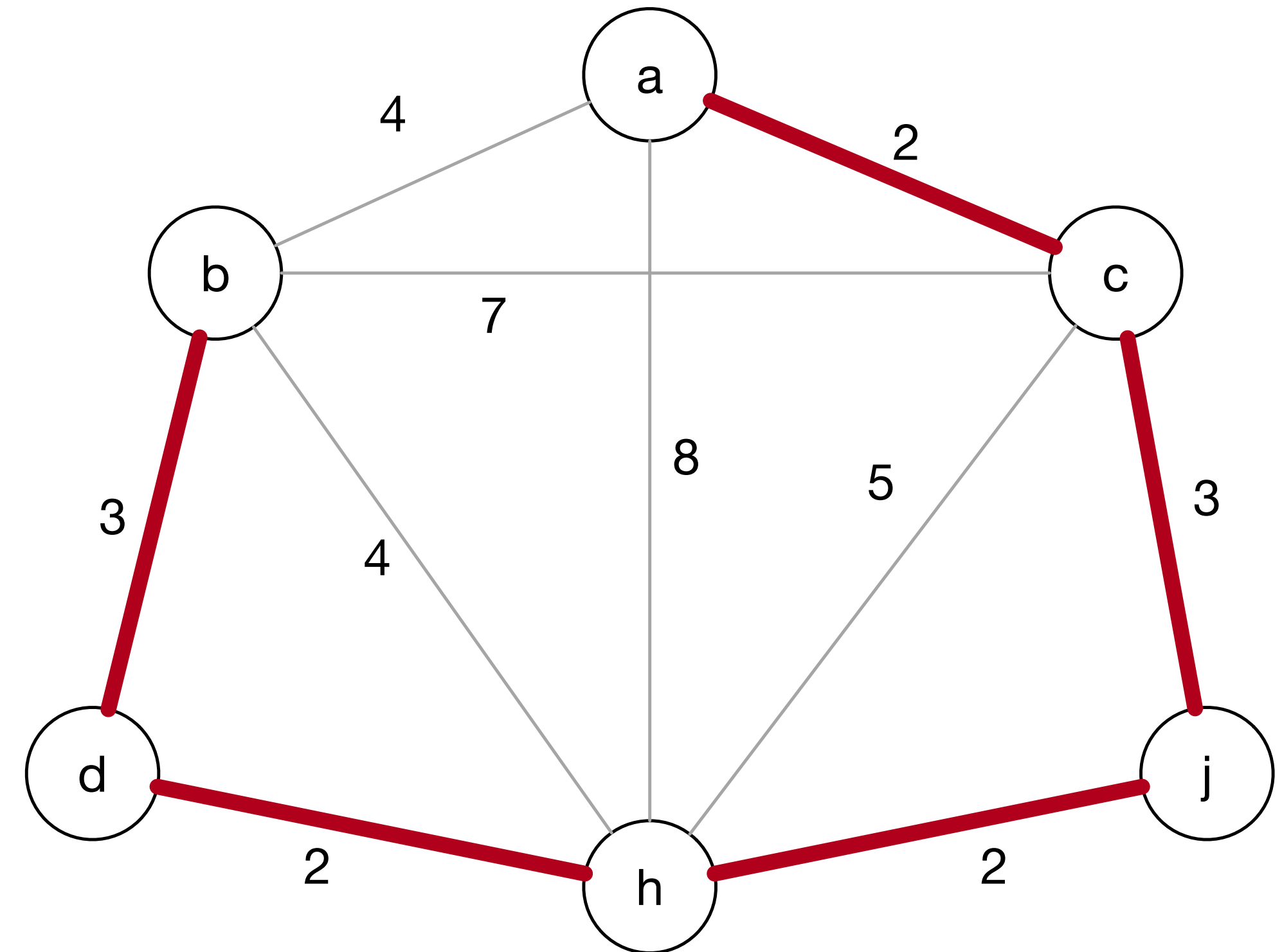
$$\operatorname{argmin} \sum_{(u,v) \in E'} w_{u,v}$$

Minimum spanning tree

$$\sum_{(u,v) \in E'} w_{u,v} = 12$$



$$G = (V, E)$$



$$G' = (V, E') \text{ with } E' \subseteq E$$

Minimum spanning tree

Some areas of application

- Telecommunications
- Taxonomy / phylogeny
- Computer networks
- Electronic circuit design
- Computer vision
- Process control
- more...

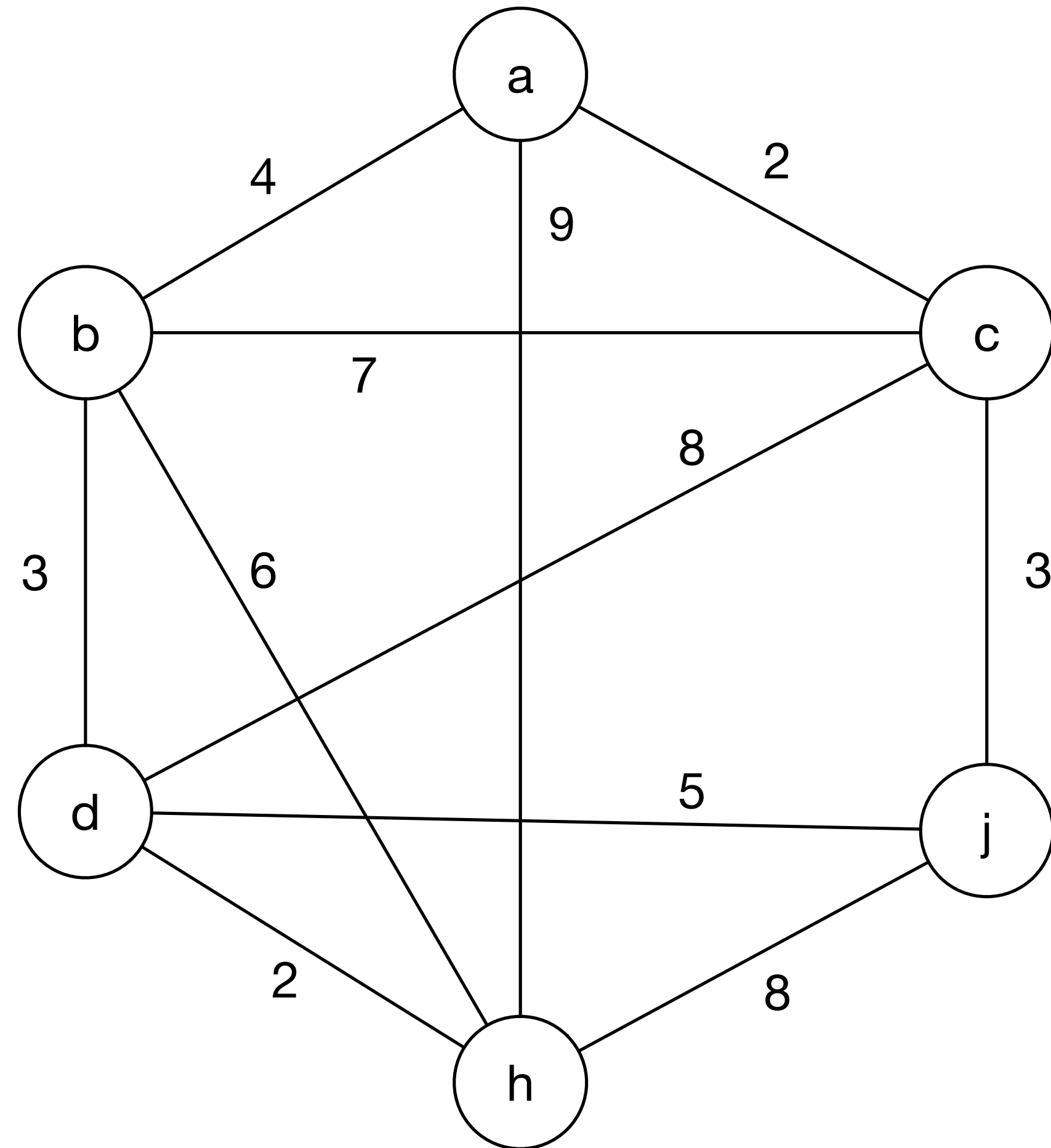
Minimum spanning tree

Prim's algorithm, given some connected graph, $G = (V, E)$

```
function prims(G)
  let  $E' = \{\}$ 
  pick any starting node,  $s$ , in  $V$ 
  let  $V' = \{s\}$ 
  while  $|V'| < |V|$ 
    find,  $e$ , the minimum weight edge with exactly one endpoint in  $V'$ 
    add this edge to  $E'$ 
    add the endpoint of  $e$  not in  $V'$  to  $V'$ 
  return  $(V', E')$ 
```

Minimum spanning tree

Prim's algorithm



$$V = \{a, b, c, d, j, h\}$$

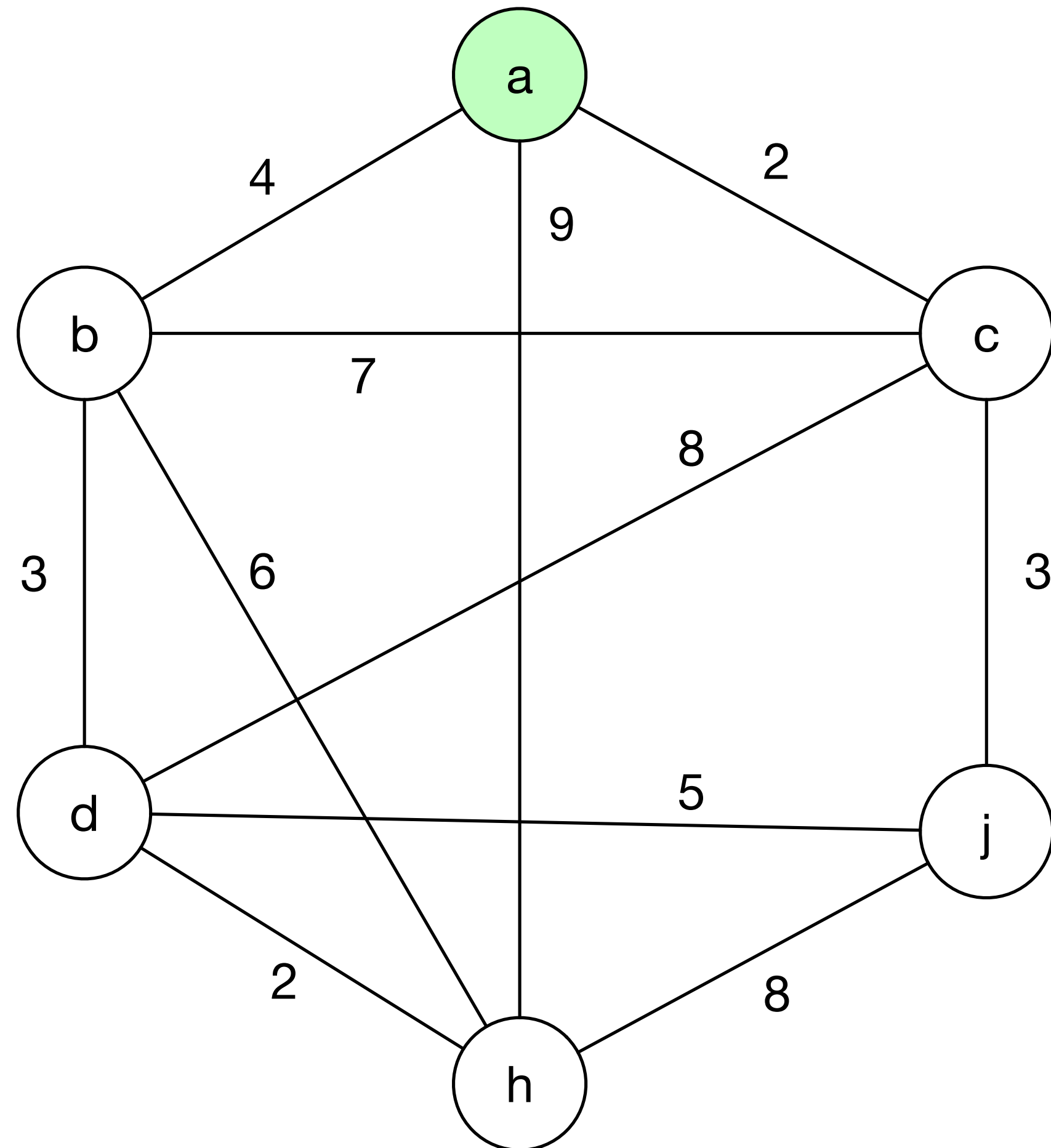
$$E = \{ab, ac, ah, bc, bd, bh, cj, cd, dj, dh, jh\}$$

$$V' = \{\}$$

$$E' = \{\}$$

Minimum spanning tree

Prim's algorithm



$$V = \{a, b, c, d, j, h\}$$

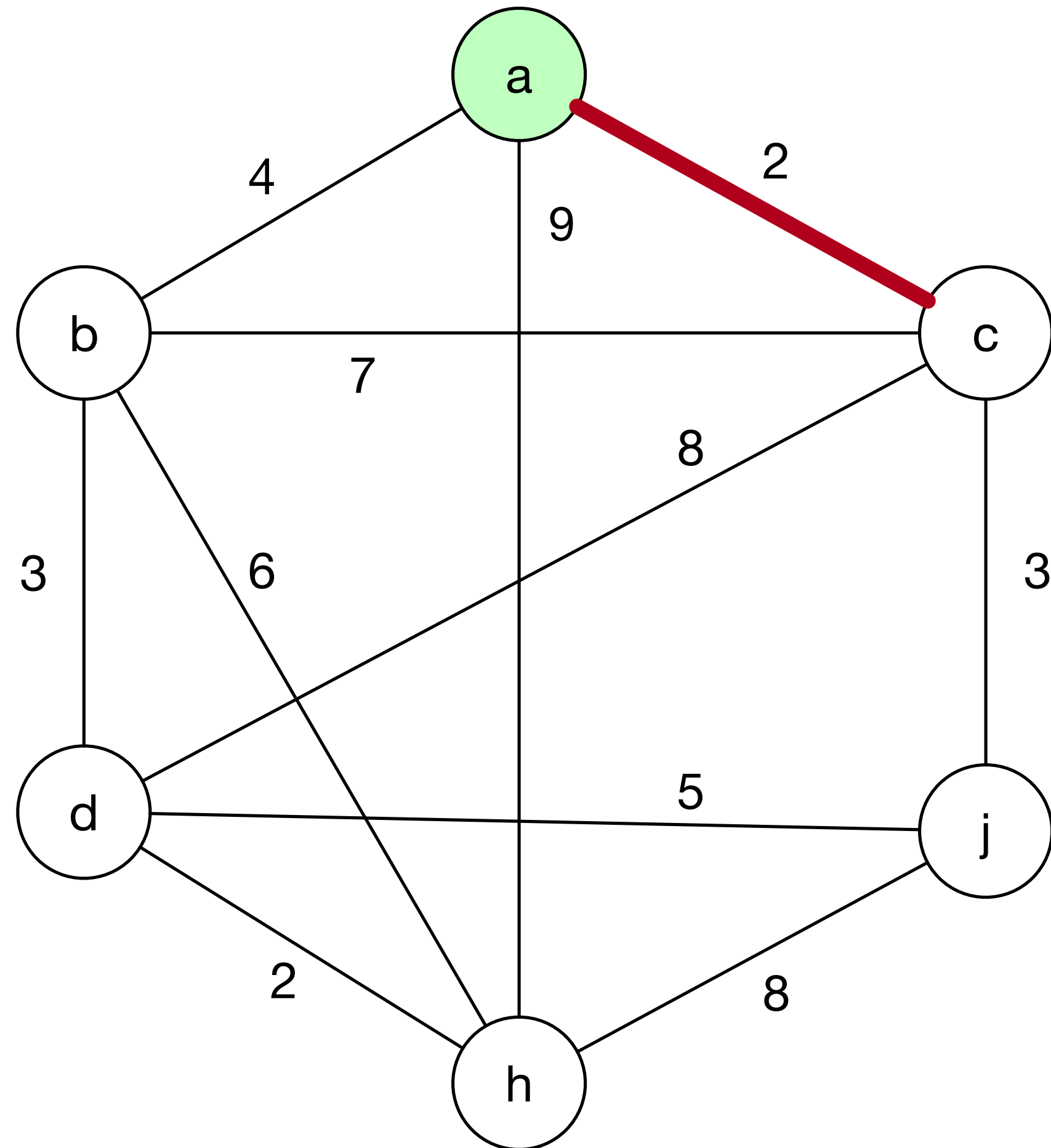
$$E = \{ab, ac, ah, bc, bd, bh, cj, cd, dj, dh, jh\}$$

$$V' = \{a\}$$

$$E' = \{\}$$

Minimum spanning tree

Prim's algorithm



$$V = \{a, b, c, d, j, h\}$$

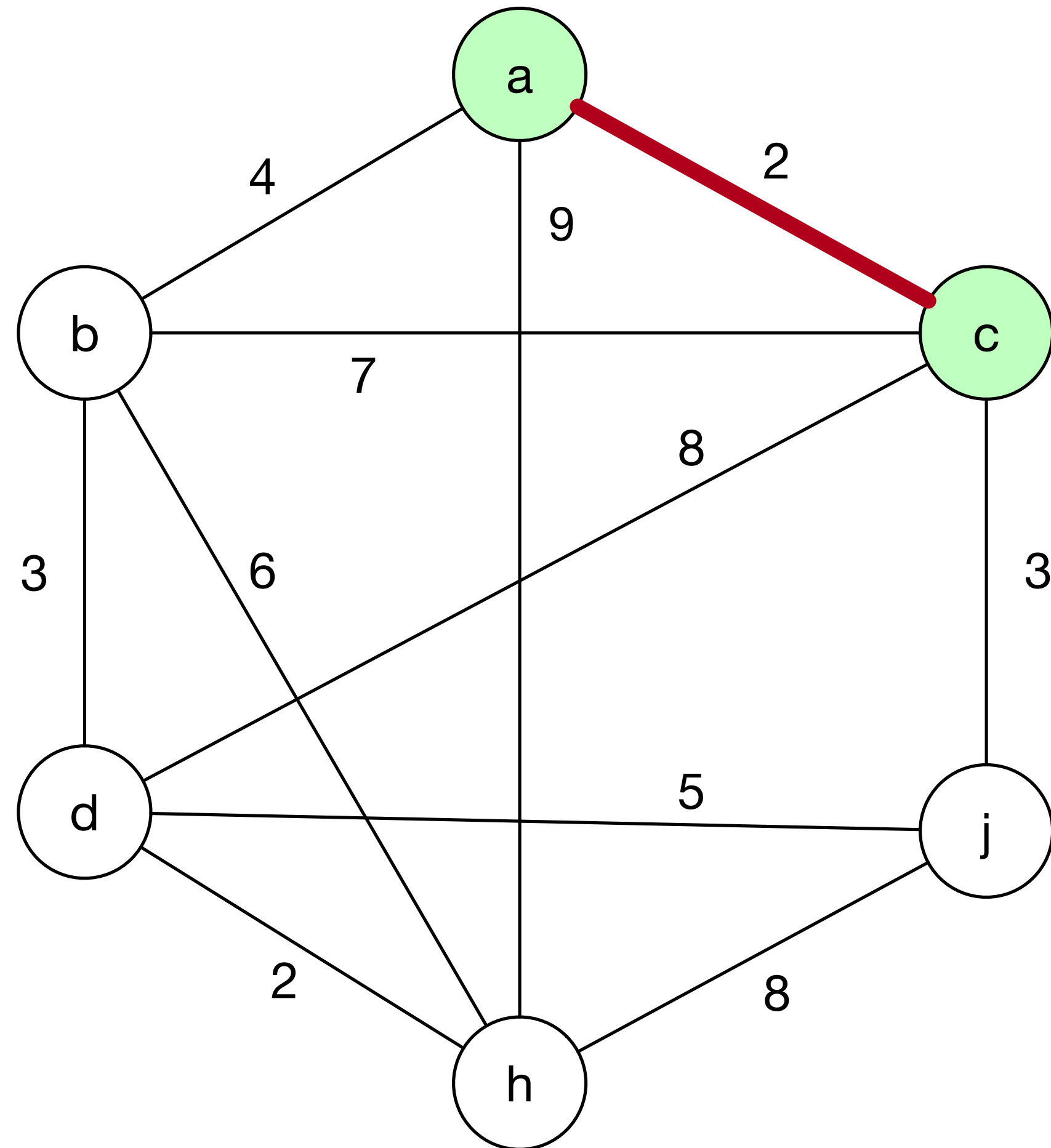
$$E = \{ab, ac, ah, bc, bd, bh, cj, cd, dj, dh, jh\}$$

$$V' = \{a\}$$

$$E' = \{ac\}$$

Minimum spanning tree

Prim's algorithm



$$V = \{a, b, c, d, j, h\}$$

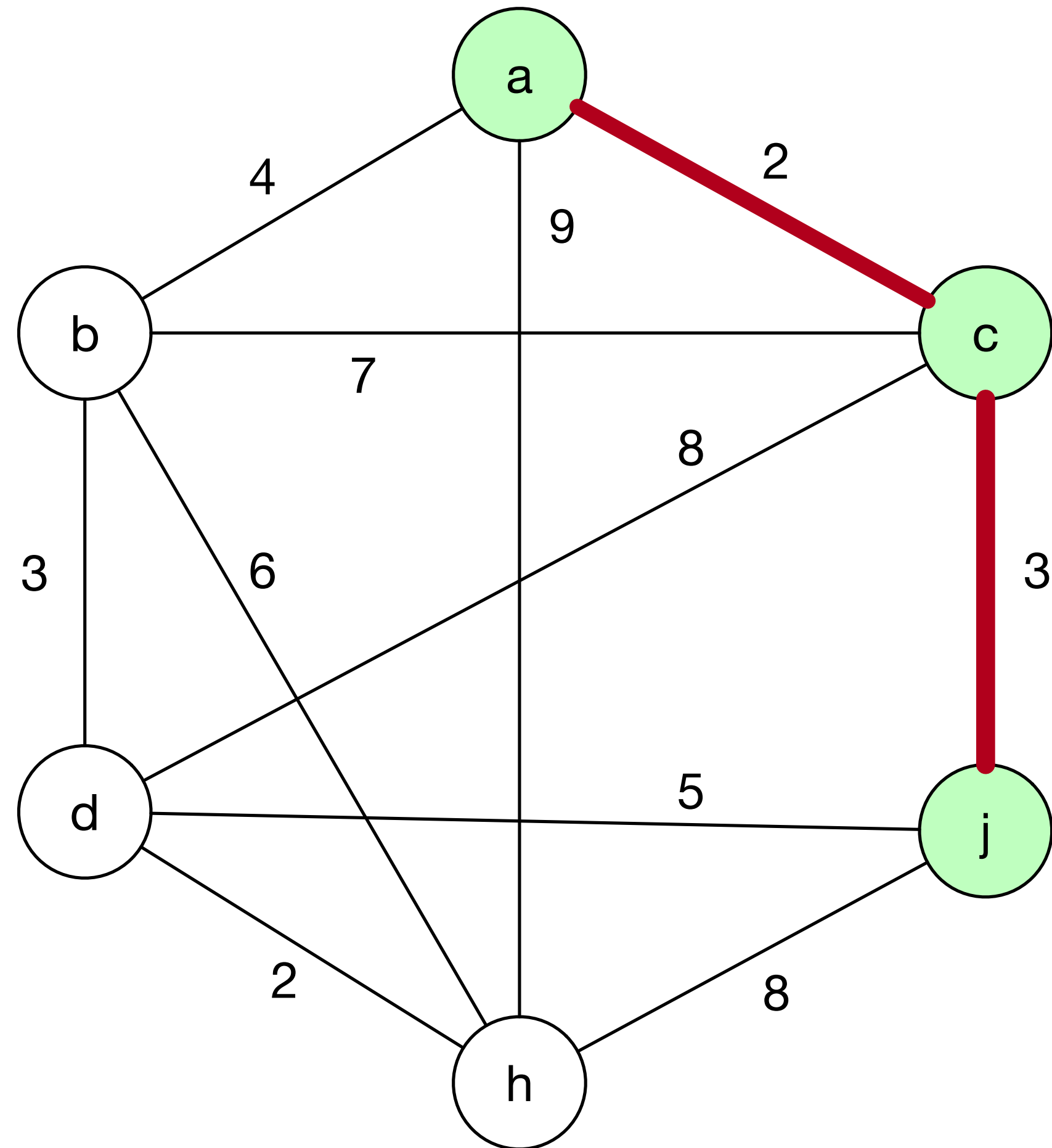
$$E = \{ab, ac, ah, bc, bd, bh, cj, cd, dj, dh, jh\}$$

$$V' = \{a, c\}$$

$$E' = \{ac\}$$

Minimum spanning tree

Prim's algorithm



$$V = \{a, b, c, d, j, h\}$$

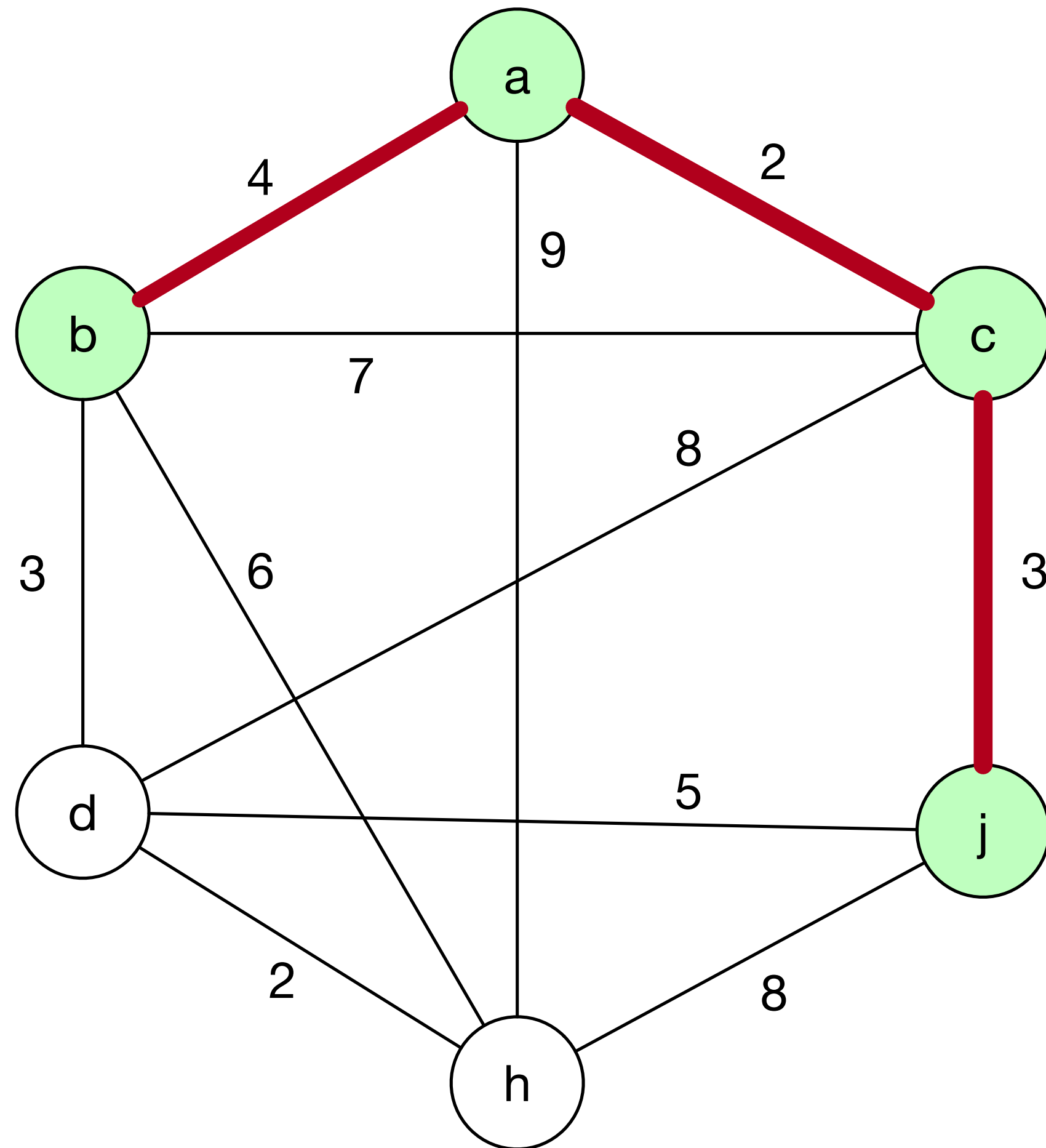
$$E = \{ab, ac, ah, bc, bd, bh, cj, cd, dj, dh, jh\}$$

$$V' = \{a, c, j\}$$

$$E' = \{ac, cj\}$$

Minimum spanning tree

Prim's algorithm



$$V = \{a, b, c, d, j, h\}$$

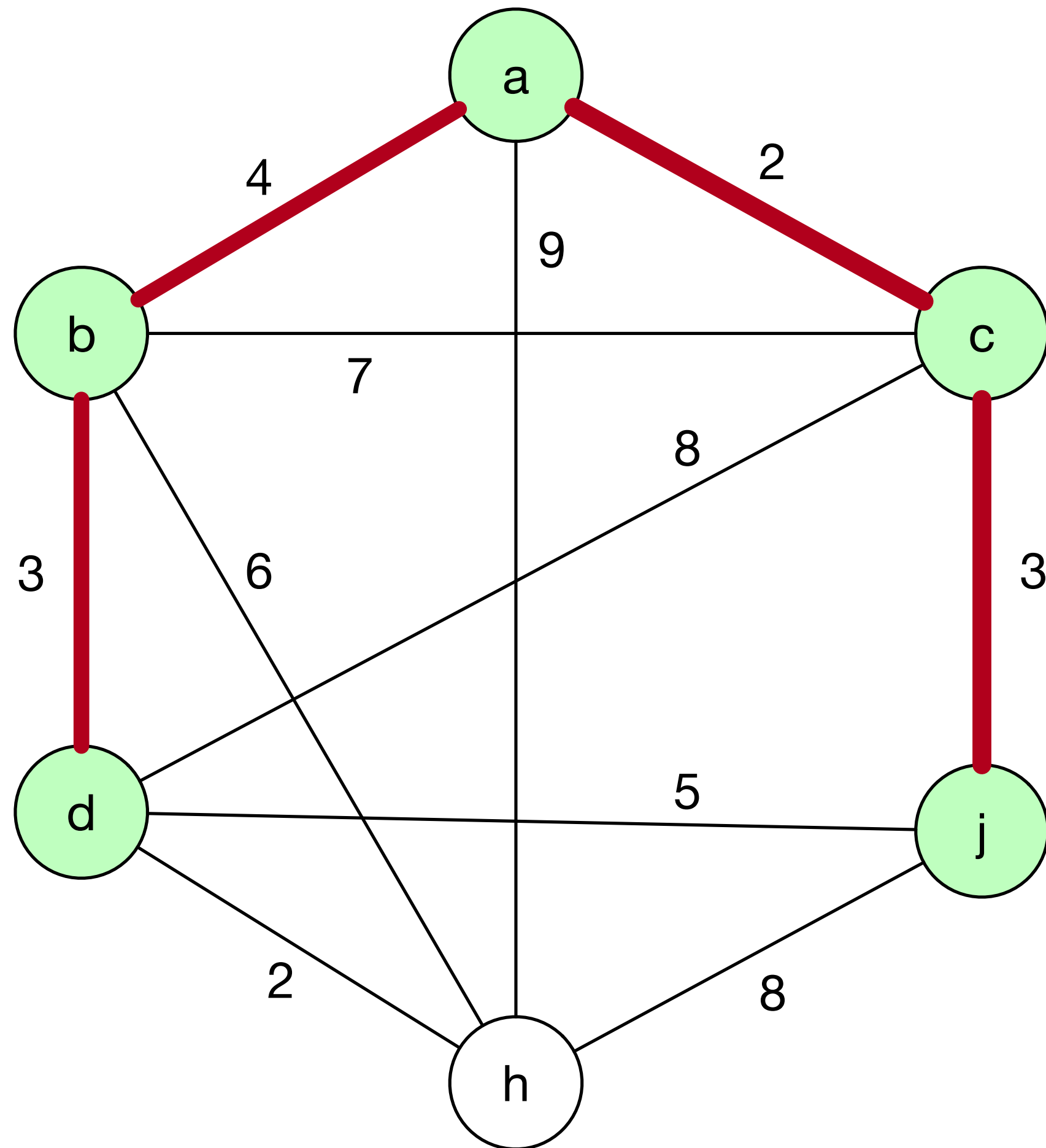
$$E = \{ab, ac, ah, bc, bd, bh, cj, cd, dj, dh, jh\}$$

$$V' = \{a, c, j, b\}$$

$$E' = \{ac, cj, ab\}$$

Minimum spanning tree

Prim's algorithm



$$V = \{a, b, c, d, j, h\}$$

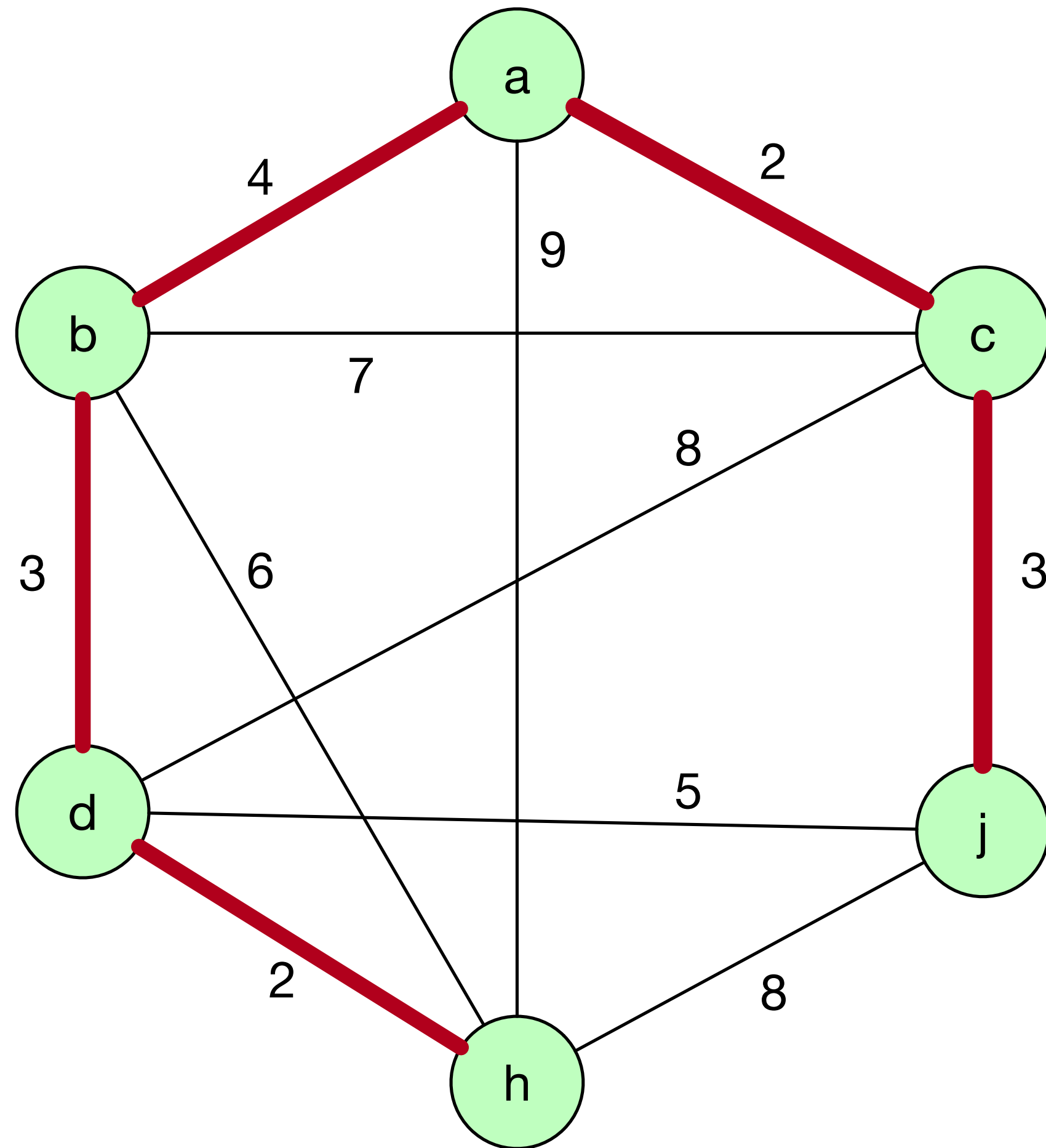
$$E = \{ab, ac, ah, bc, bd, bh, cj, cd, dj, dh, jh\}$$

$$V' = \{a, c, j, b, d\}$$

$$E' = \{ac, cj, ab, bd\}$$

Minimum spanning tree

Prim's algorithm



$$V = \{a, b, c, d, j, h\}$$

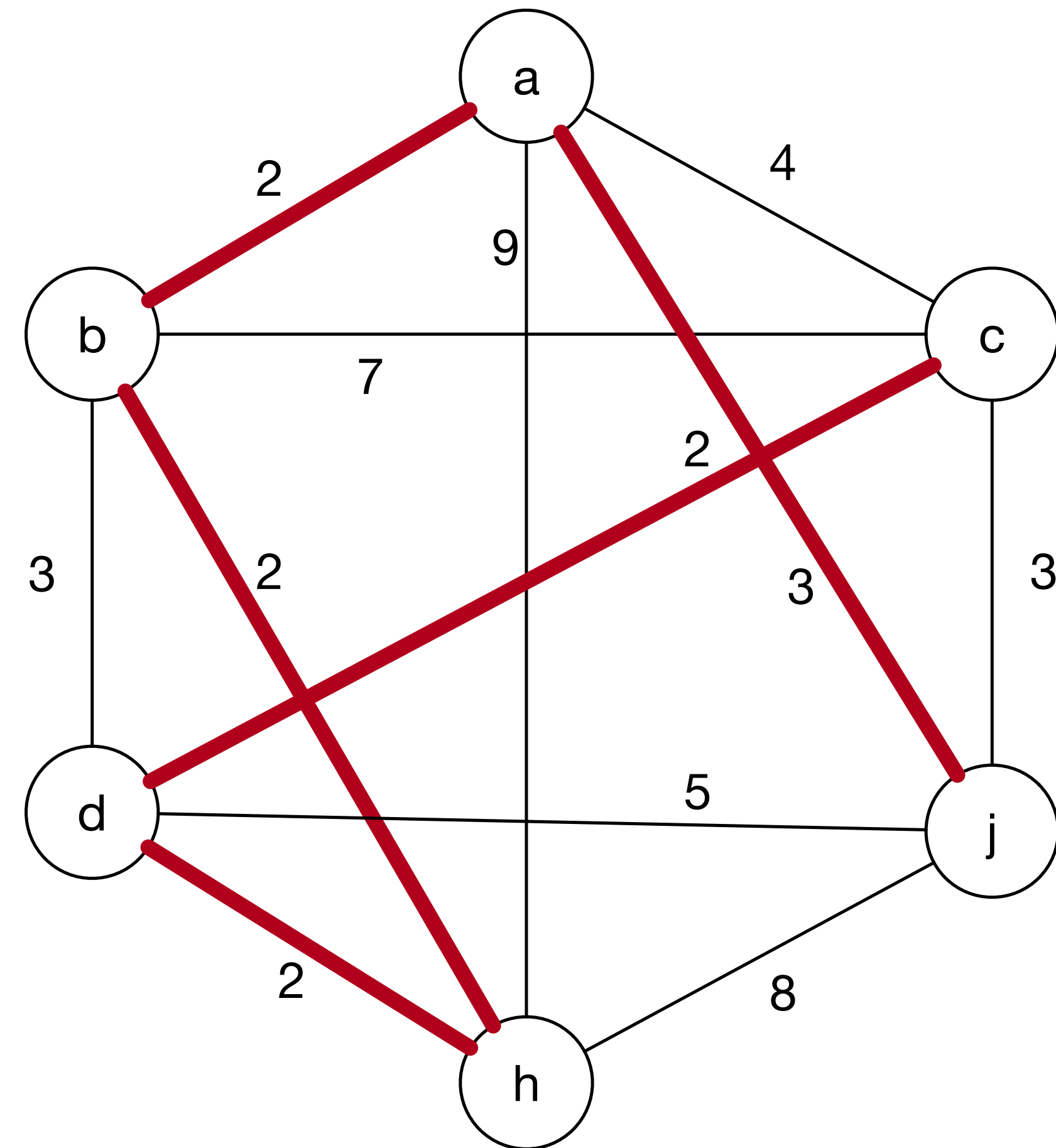
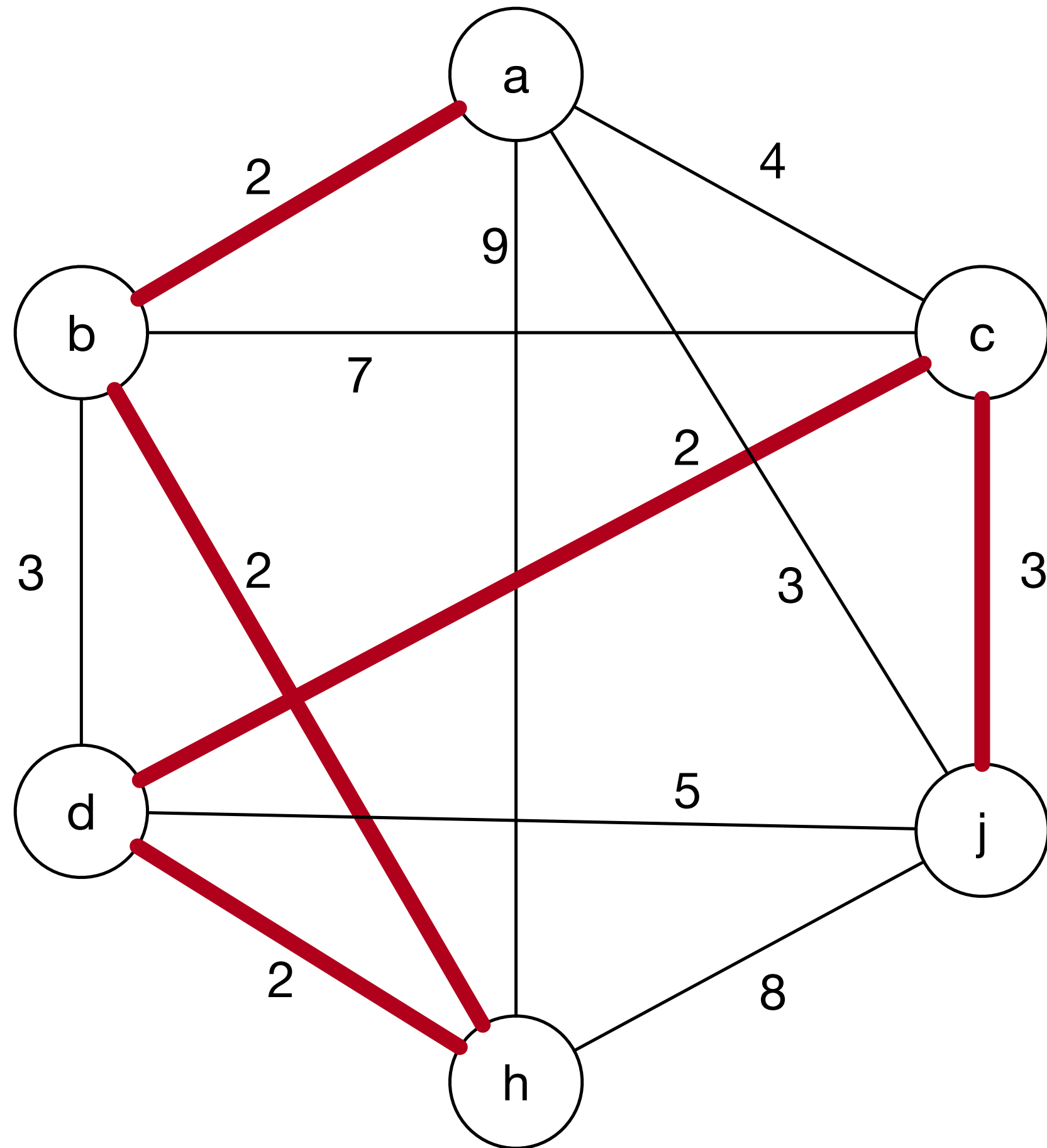
$$E = \{ab, ac, ah, bc, bd, bh, cj, cd, dj, dh, jh\}$$

$$V' = \{a, c, j, b, d, h\}$$

$$E' = \{ac, cj, ab, bd, dh\}$$

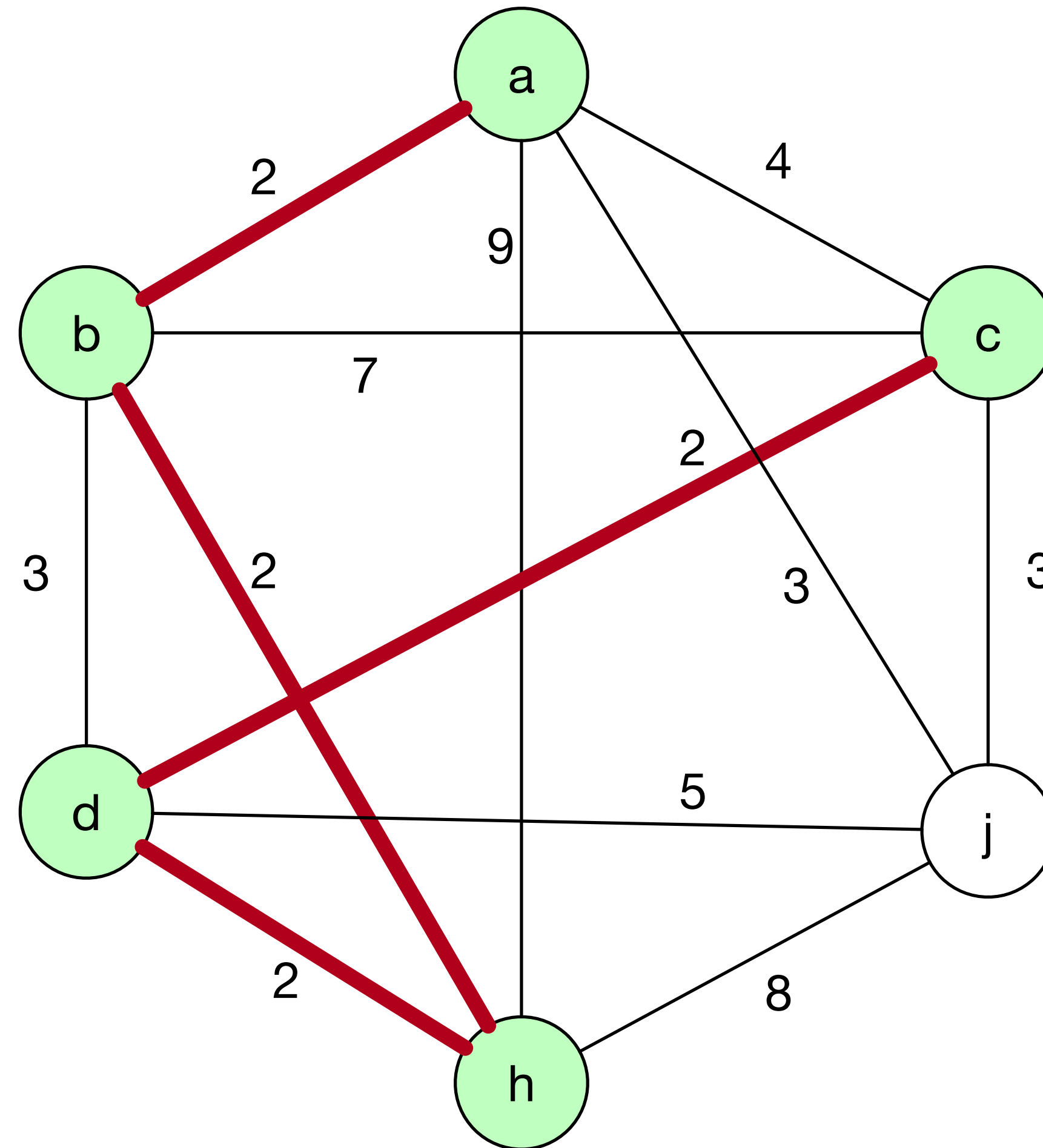
Minimum spanning tree

Prim's algorithm



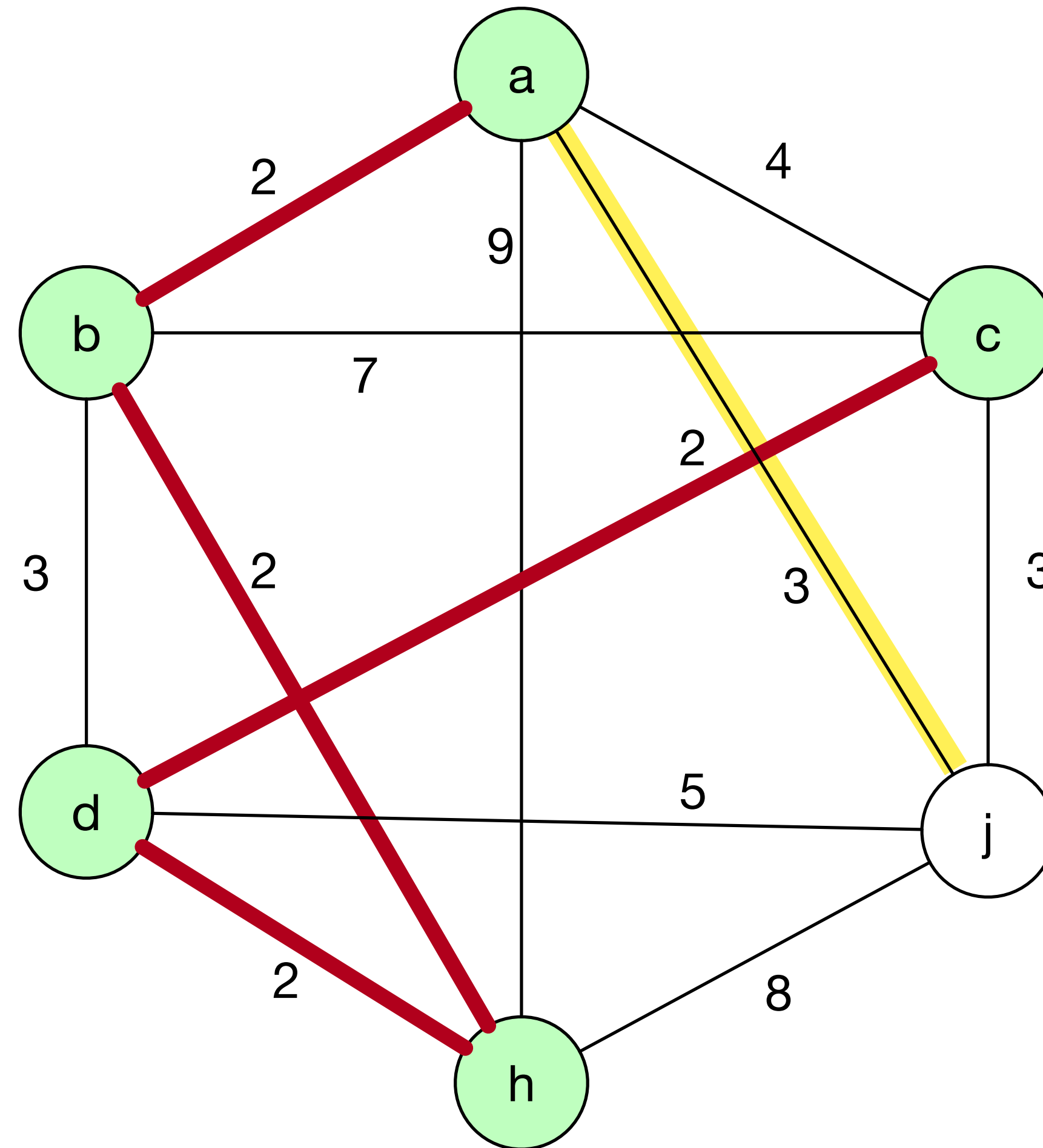
Minimum spanning tree

Prim's algorithm



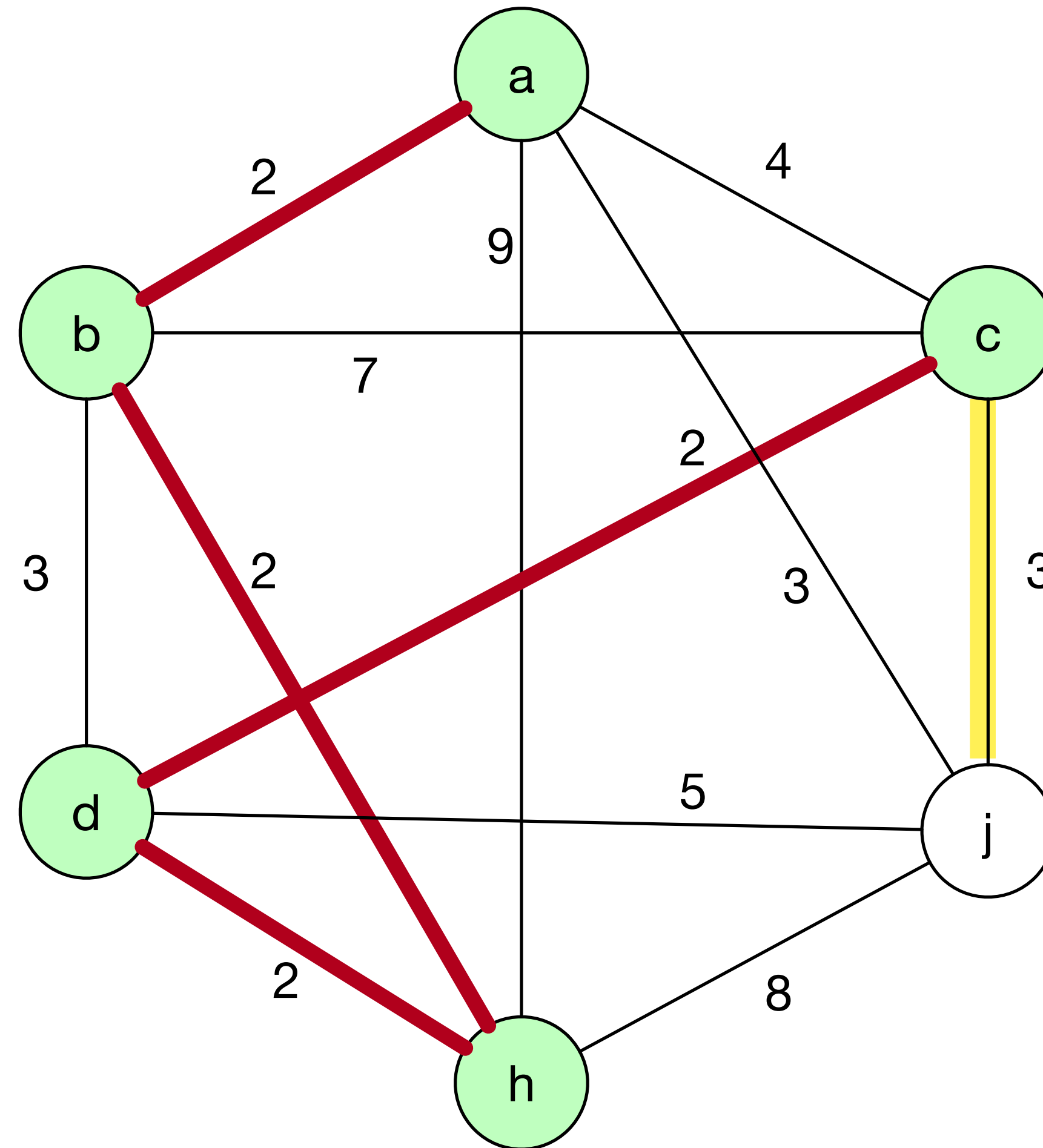
Minimum spanning tree

Prim's algorithm



Minimum spanning tree

Prim's algorithm

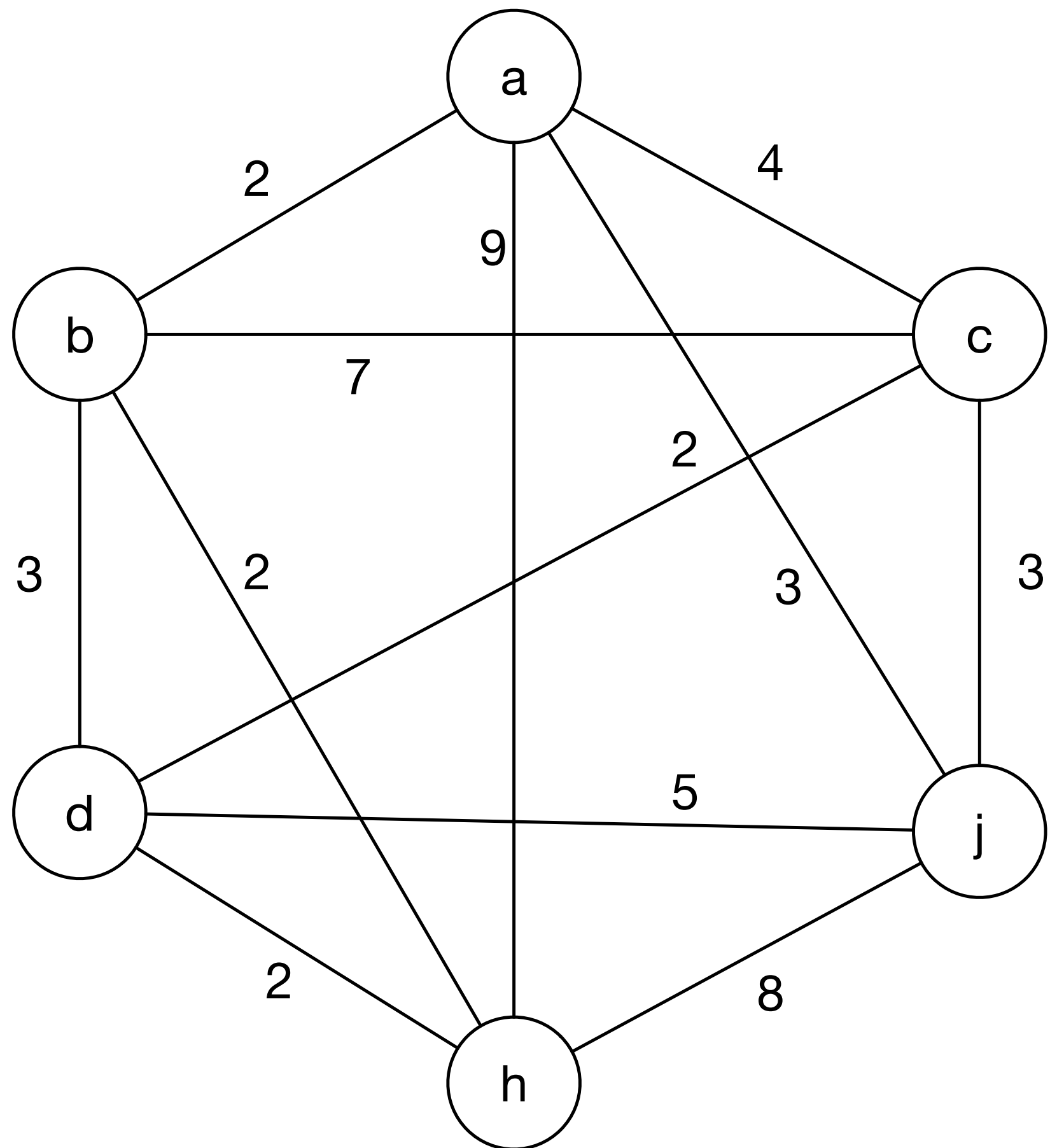


Minimum spanning tree

Prim's algorithm

$$V' = \{\}$$

$$E' = \{\}$$



	a	b	c	d	j	h
a	0	2	4	0	3	9
b		0	7	3	0	2
c			0	2	3	0
d				0	5	2
j					0	8
h						0

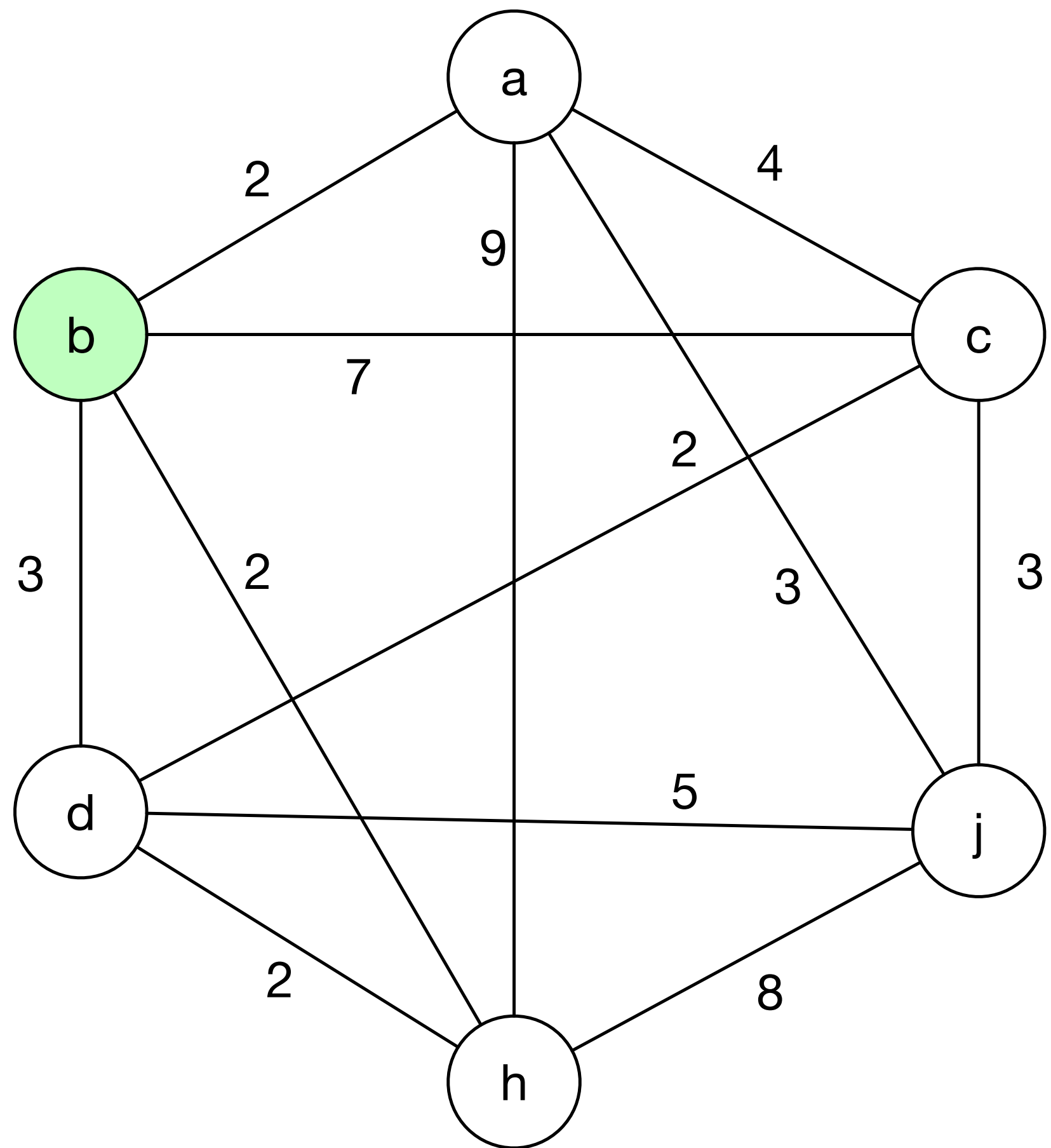
	lowest in V'	weight
a	-	-
b	-	-
c	-	-
d	-	-
j	-	-
h	-	-

Minimum spanning tree

Prim's algorithm

$$V' = \{b\}$$

$$E' = \{\}$$



	a	b	c	d	j	h
a	0	2	4	0	3	9
b		0	7	3	0	2
c			0	2	3	0
d				0	5	2
j					0	8
h						0

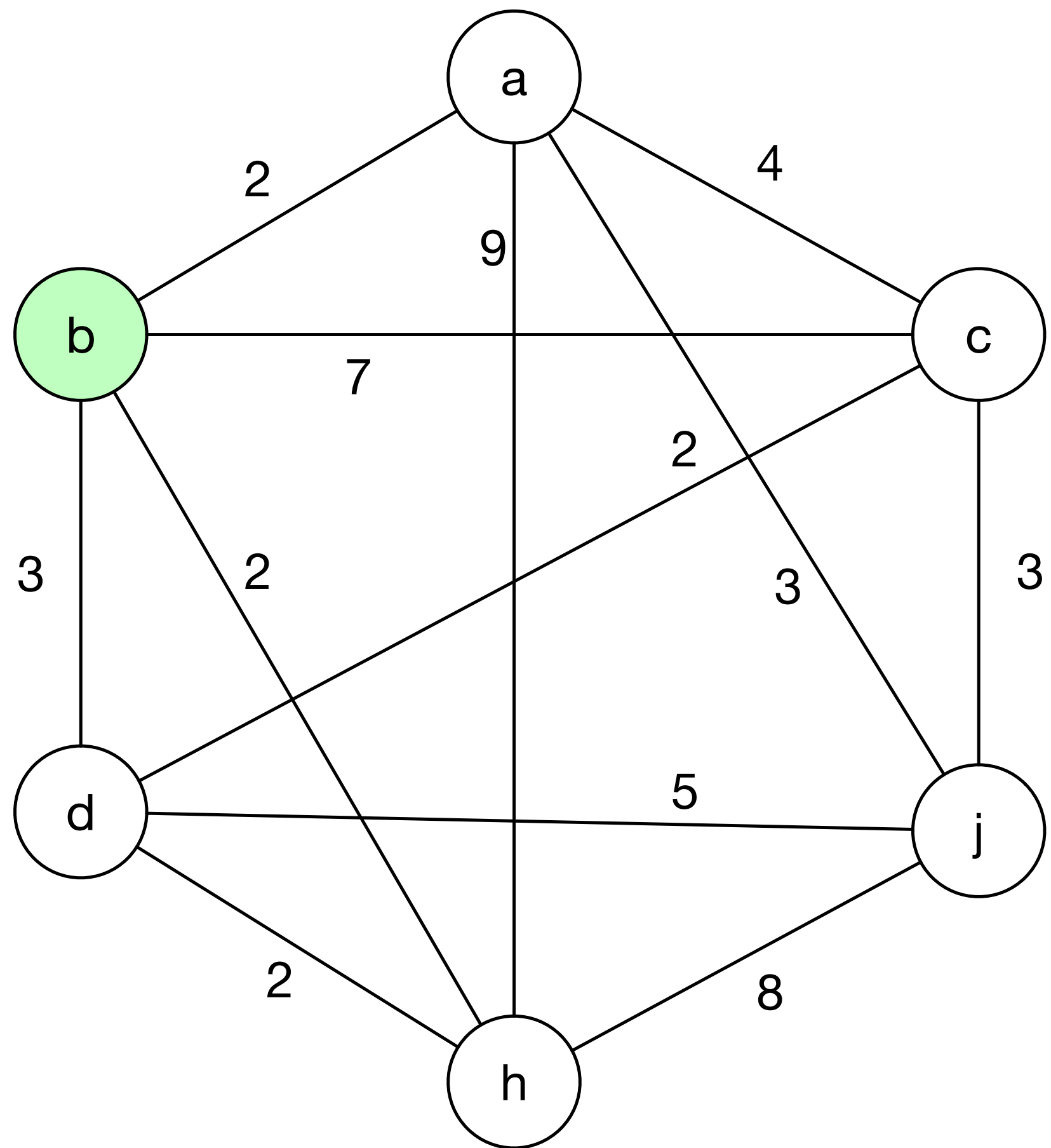
	lowest in V'	weight
a	b	2
b	-	-
c	b	7
d	b	3
j	-	-
h	b	2

Minimum spanning tree

Prim's algorithm

$$V' = \{b\}$$

$$E' = \{\}$$



	a	b	c	d	j	h
a	0	2	4	0	3	9
b		0	7	3	0	2
c			0	2	3	0
d				0	5	2
j					0	8
h						0

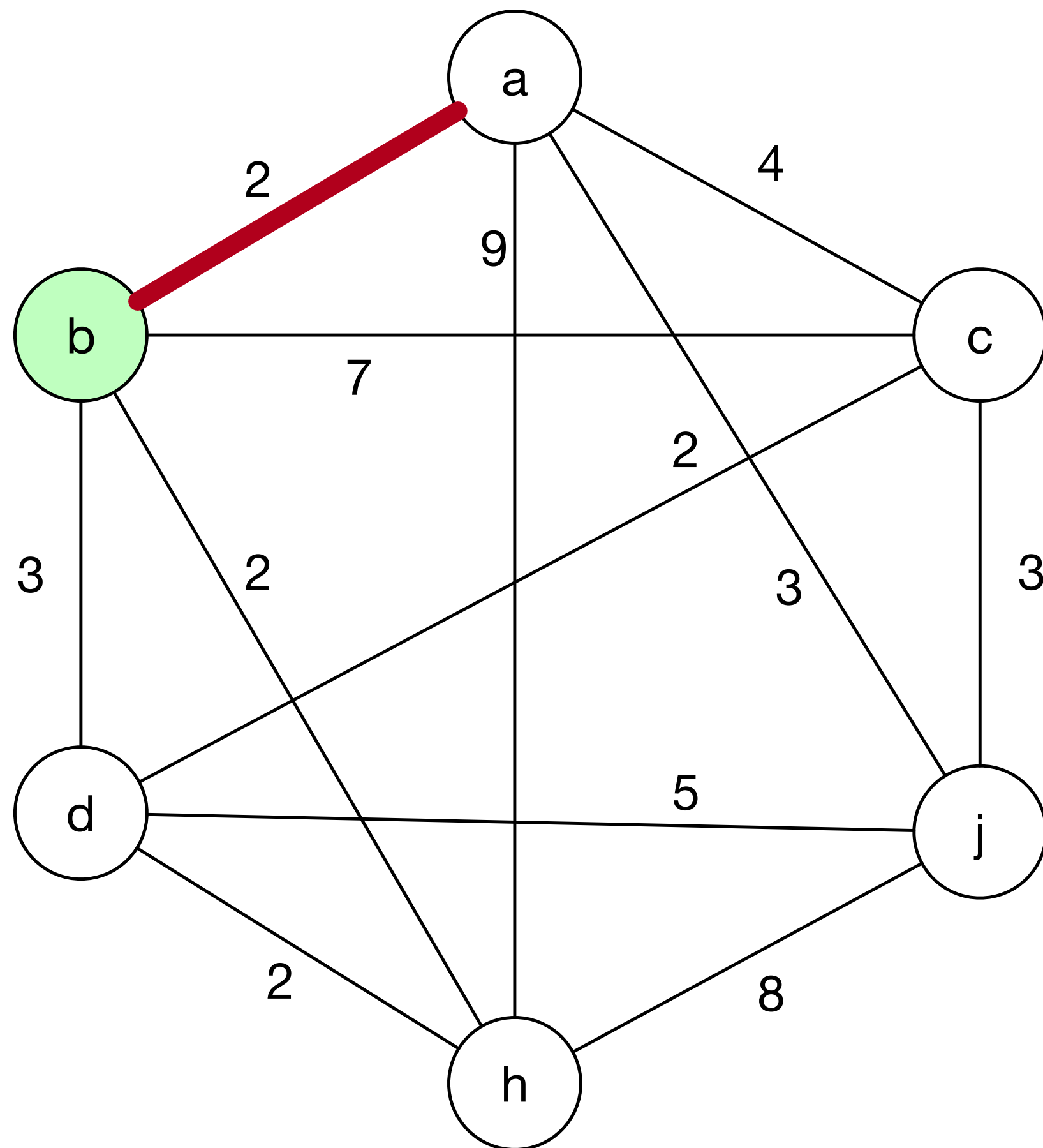
	lowest in V'	weight
a	b	2
b	-	-
c	b	7
d	b	3
j	-	-
h	b	2

Minimum spanning tree

Prim's algorithm

$$V' = \{b\}$$

$$E' = \{ab\}$$



	a	b	c	d	j	h
a	0	2	4	0	3	9
b		0	7	3	0	2
c			0	2	3	0
d				0	5	2
j					0	8
h						0

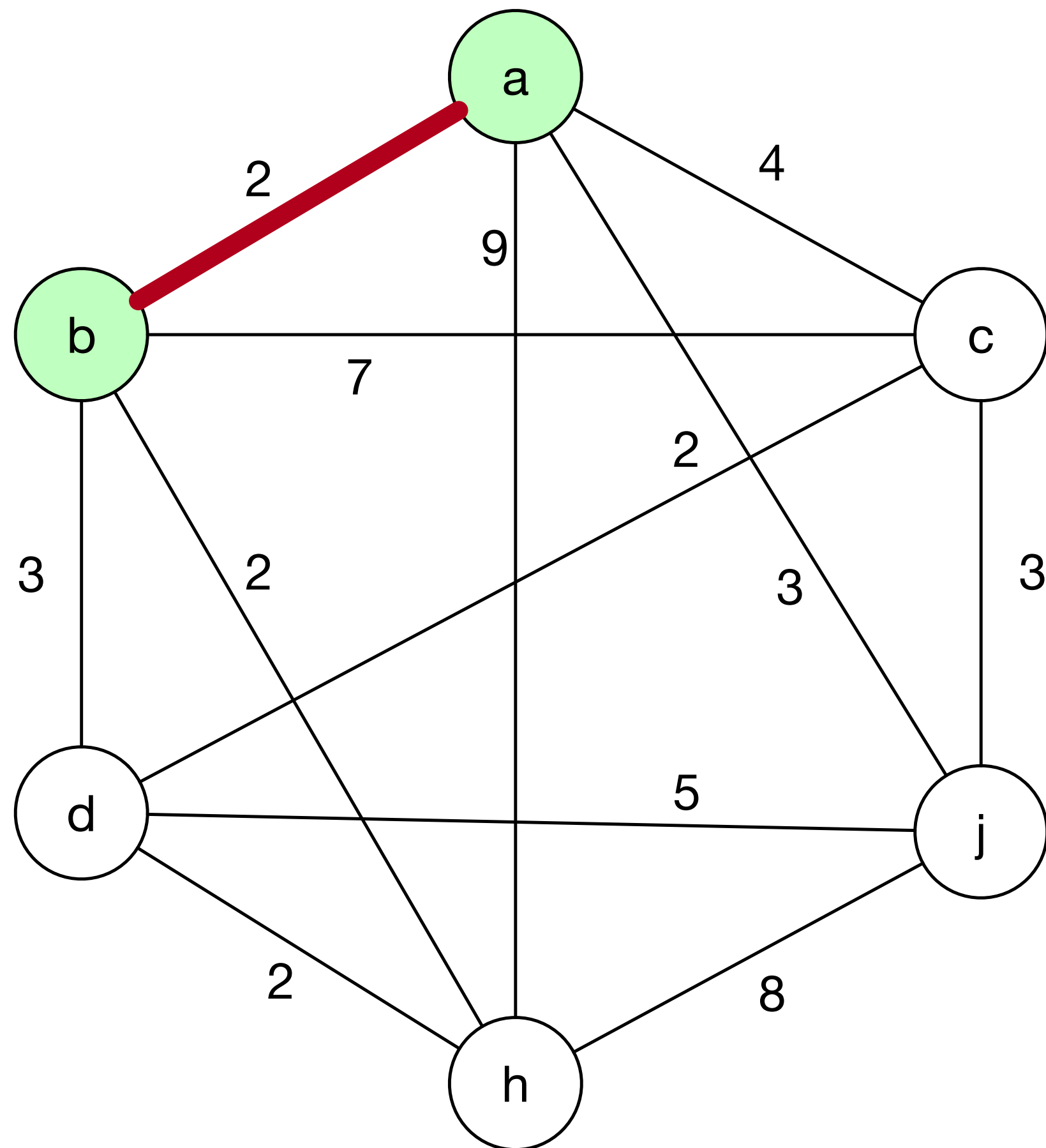
	lowest in V'	weight
a	b	2
b	-	-
c	b	7
d	b	3
j	-	-
h	b	2

Minimum spanning tree

Prim's algorithm

$$V' = \{b, a\}$$

$$E' = \{ab\}$$



	a	b	c	d	j	h
a	0	2	4	0	3	9
b		0	7	3	0	2
c			0	2	3	0
d				0	5	2
j					0	8
h						0

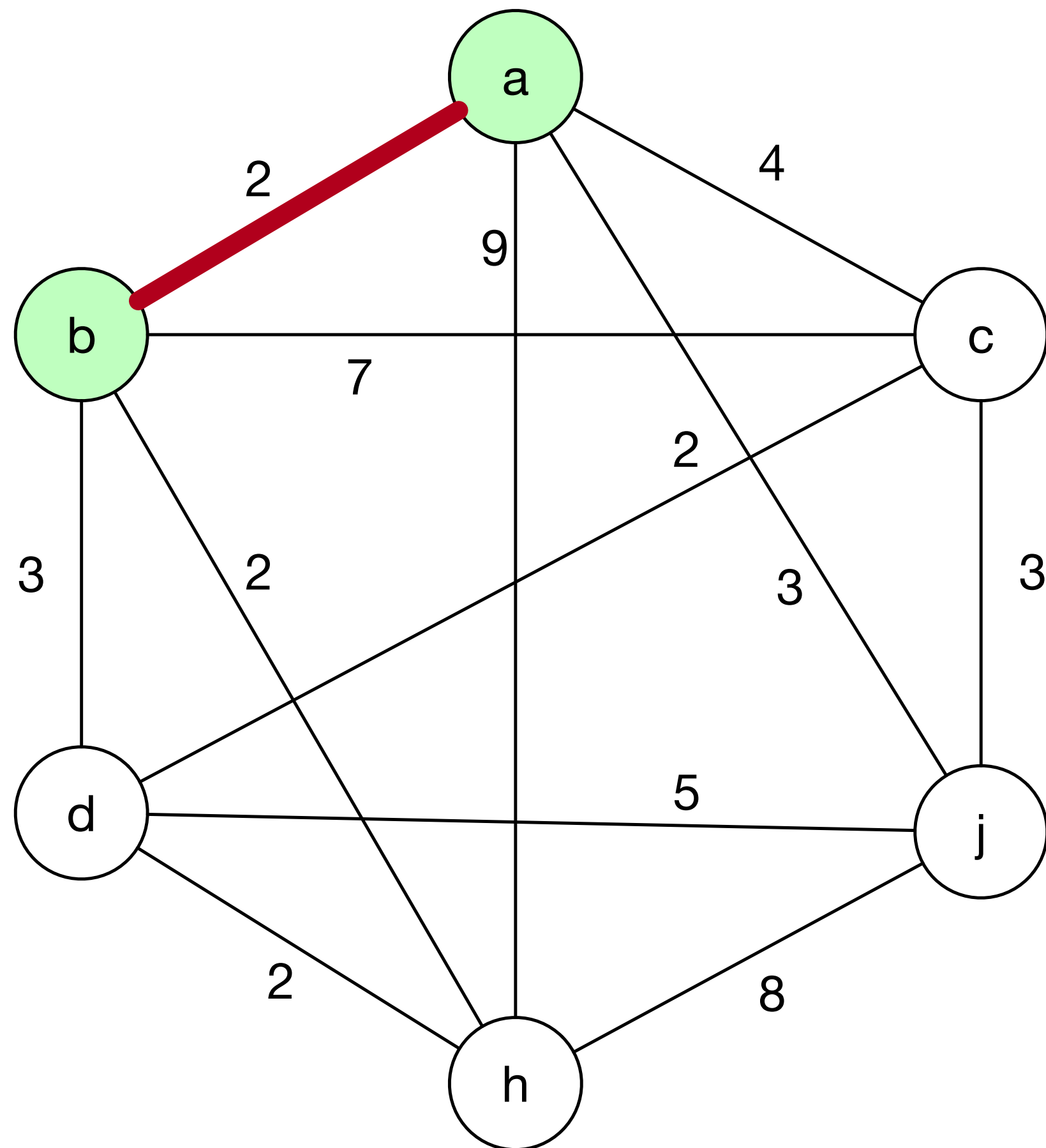
	lowest in V'	weight
a	-	-
b	-	-
c	b	7
d	b	3
j	-	-
h	b	2

Minimum spanning tree

Prim's algorithm

$$V' = \{b, a\}$$

$$E' = \{ab\}$$



	a	b	c	d	j	h
a	0	2	4	0	3	9
b		0	7	3	0	2
c			0	2	3	0
d				0	5	2
j					0	8
h						0

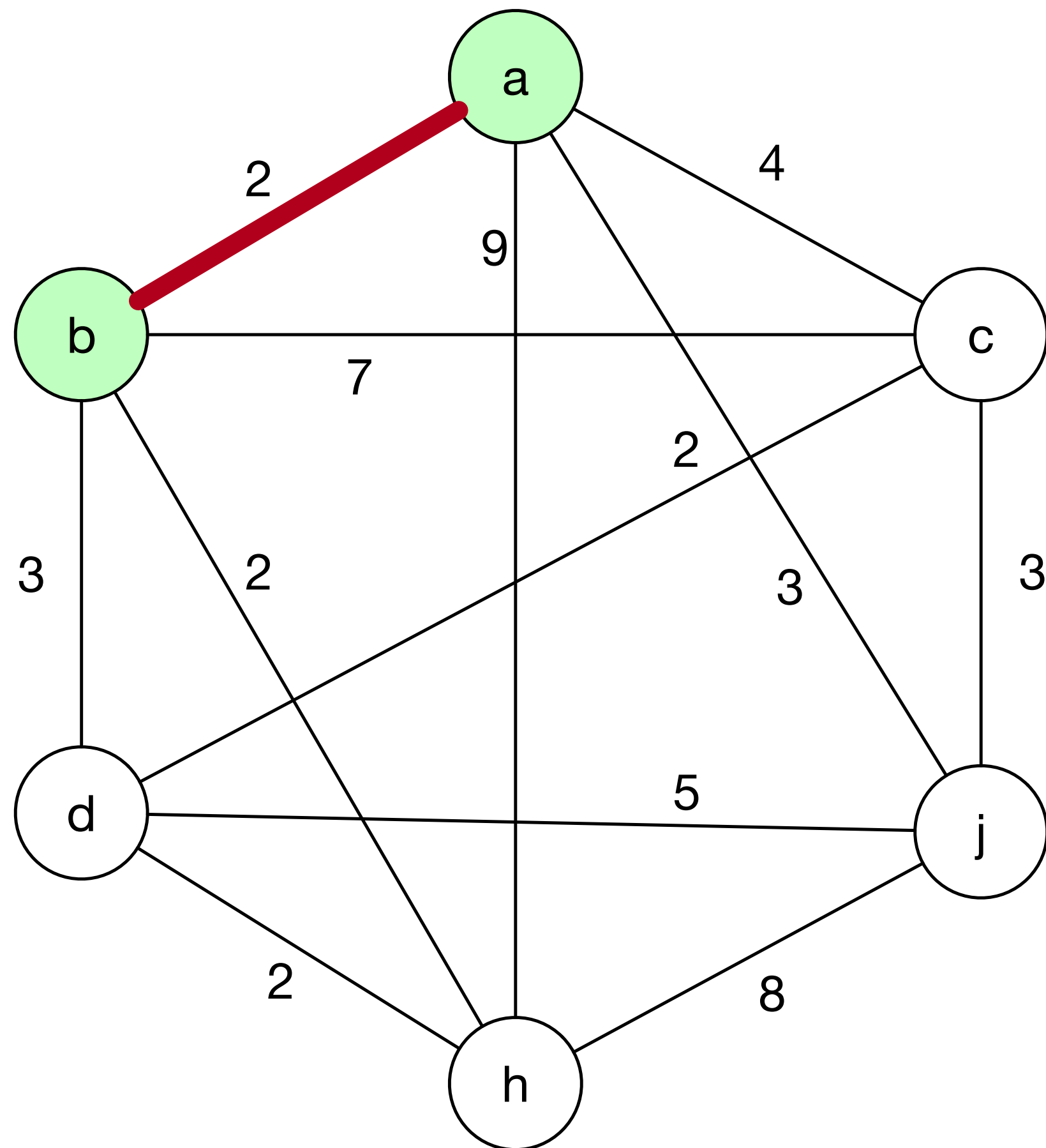
	lowest in V'	weight
a	-	-
b	-	-
c	a	4
d	b	3
j	-	-
h	b	2

Minimum spanning tree

Prim's algorithm

$$V' = \{b, a\}$$

$$E' = \{ab\}$$



	a	b	c	d	j	h
a	0	2	4	0	3	9
b		0	7	3	0	2
c			0	2	3	0
d				0	5	2
j					0	8
h						0

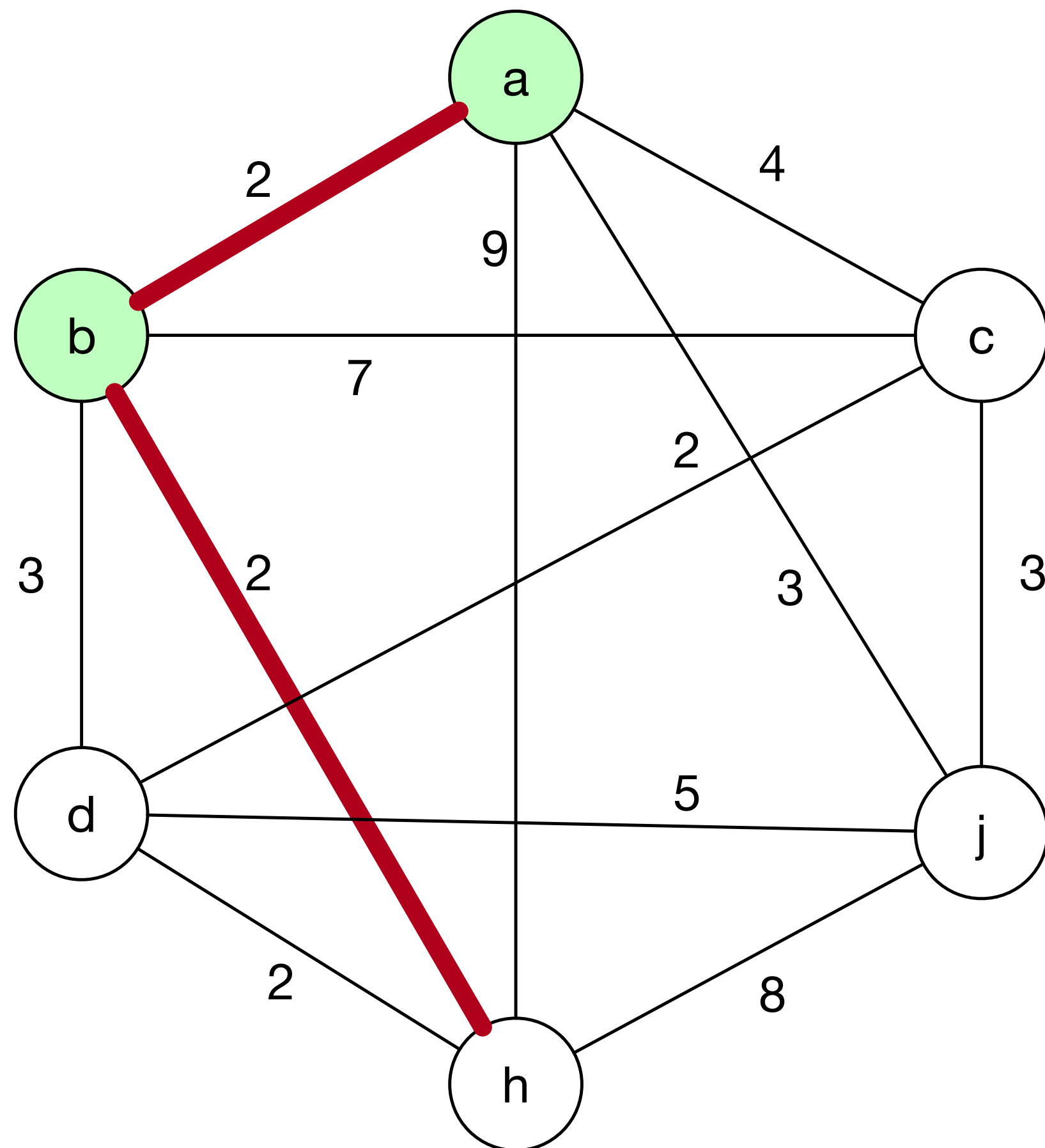
	lowest in V'	weight
a	-	-
b	-	-
c	a	4
d	b	3
j	a	3
h	b	2

Minimum spanning tree

Prim's algorithm

$$V' = \{b, a\}$$

$$E' = \{ab, bh\}$$



	a	b	c	d	j	h
a	0	2	4	0	3	9
b		0	7	3	0	2
c			0	2	3	0
d				0	5	2
j					0	8
h						0

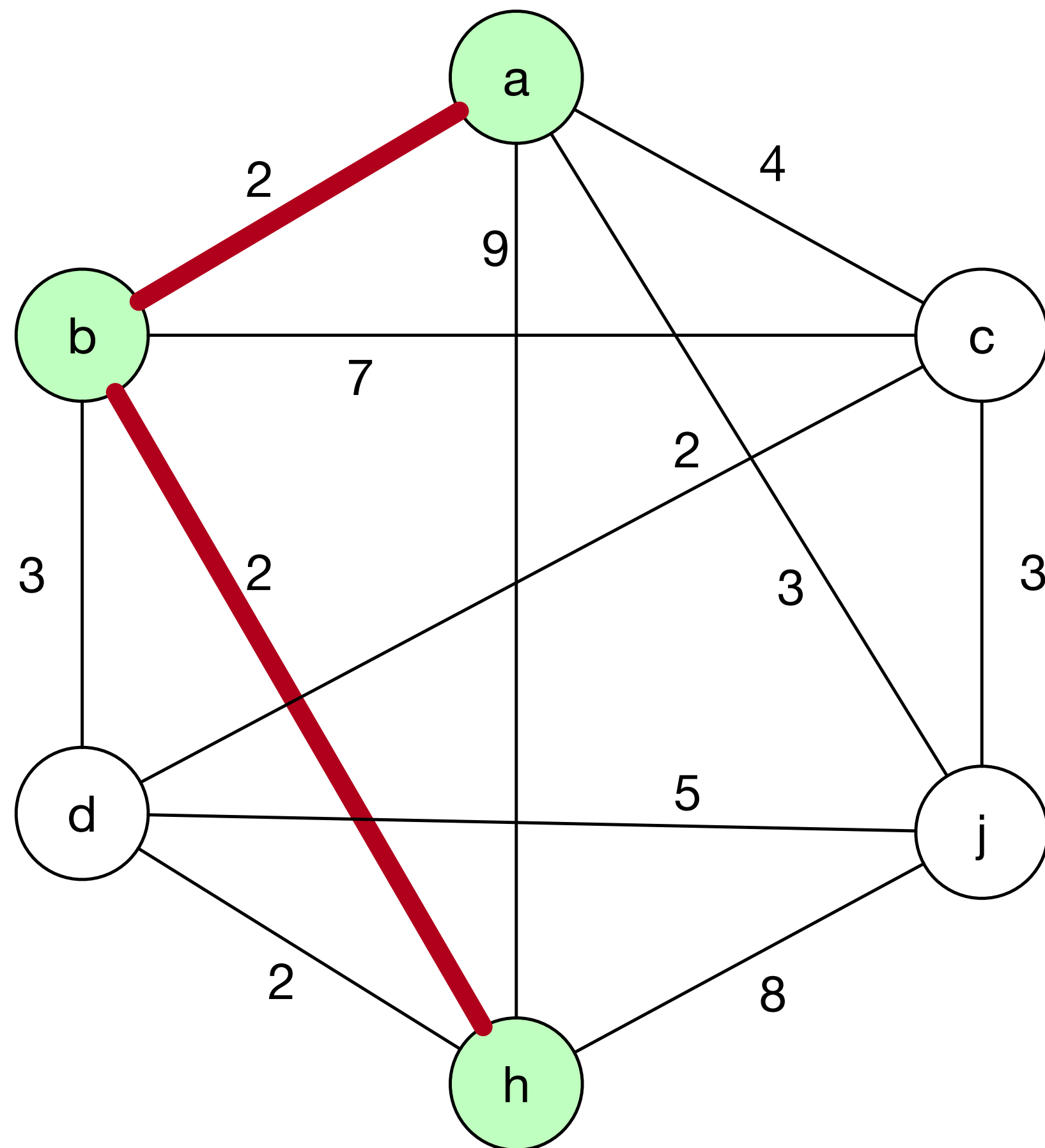
	lowest in V'	weight
a	-	-
b	-	-
c	a	4
d	b	3
j	a	3
h	b	2

Minimum spanning tree

Prim's algorithm

$$V' = \{b, a, h\}$$

$$E' = \{ab, bh\}$$



	a	b	c	d	j	h
a	0	2	4	0	3	9
b		0	7	3	0	2
c			0	2	3	0
d				0	5	2
j					0	8
h						0

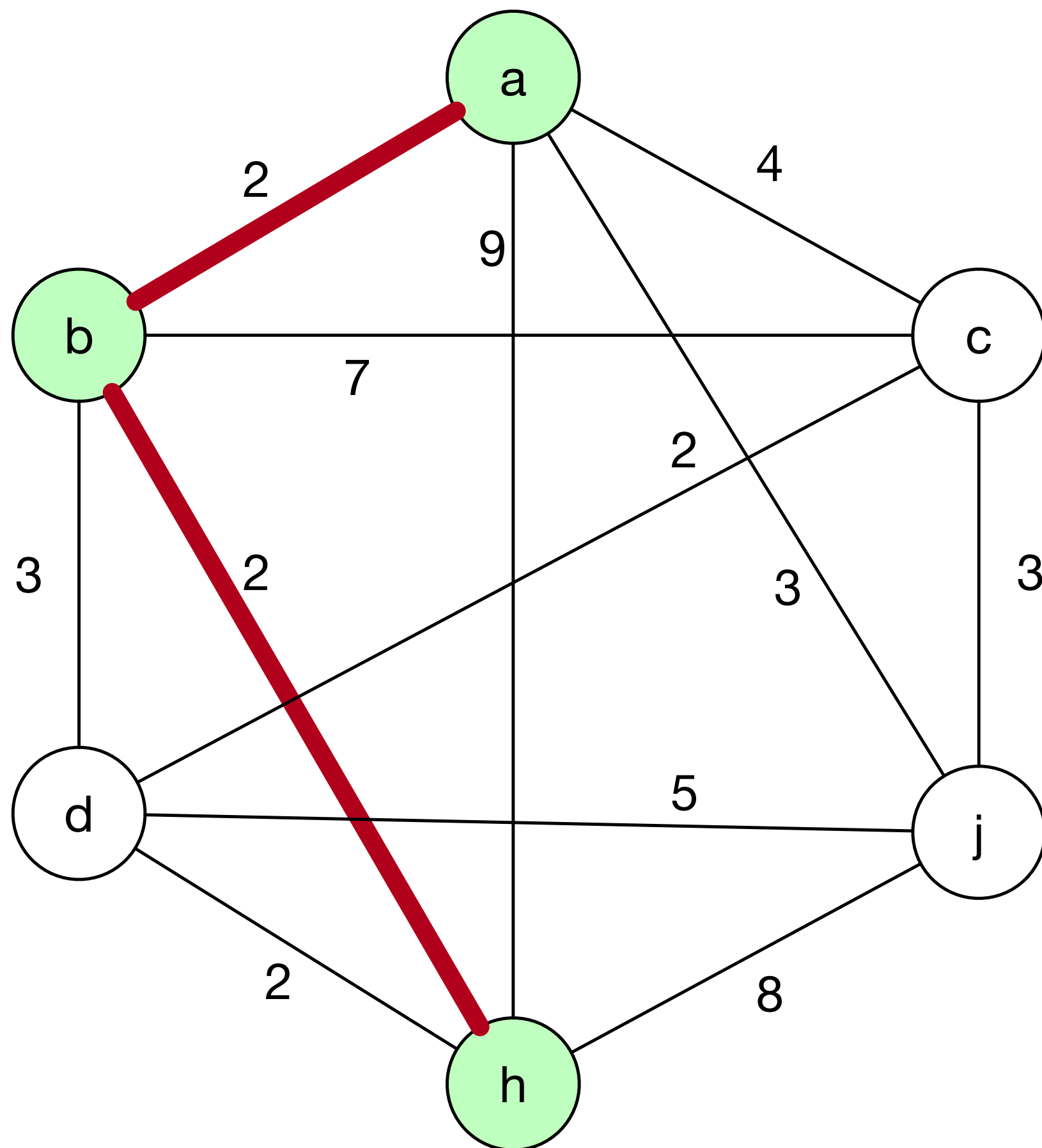
	lowest in V'	weight
a	-	-
b	-	-
c	a	4
d	b	3
j	a	3
h	b	2

Minimum spanning tree

Prim's algorithm

$$V' = \{b, a, h\}$$

$$E' = \{ab, bh\}$$



	a	b	c	d	j	h
a	0	2	4	0	3	9
b		0	7	3	0	2
c			0	2	3	0
d				0	5	2
j					0	8
h						0

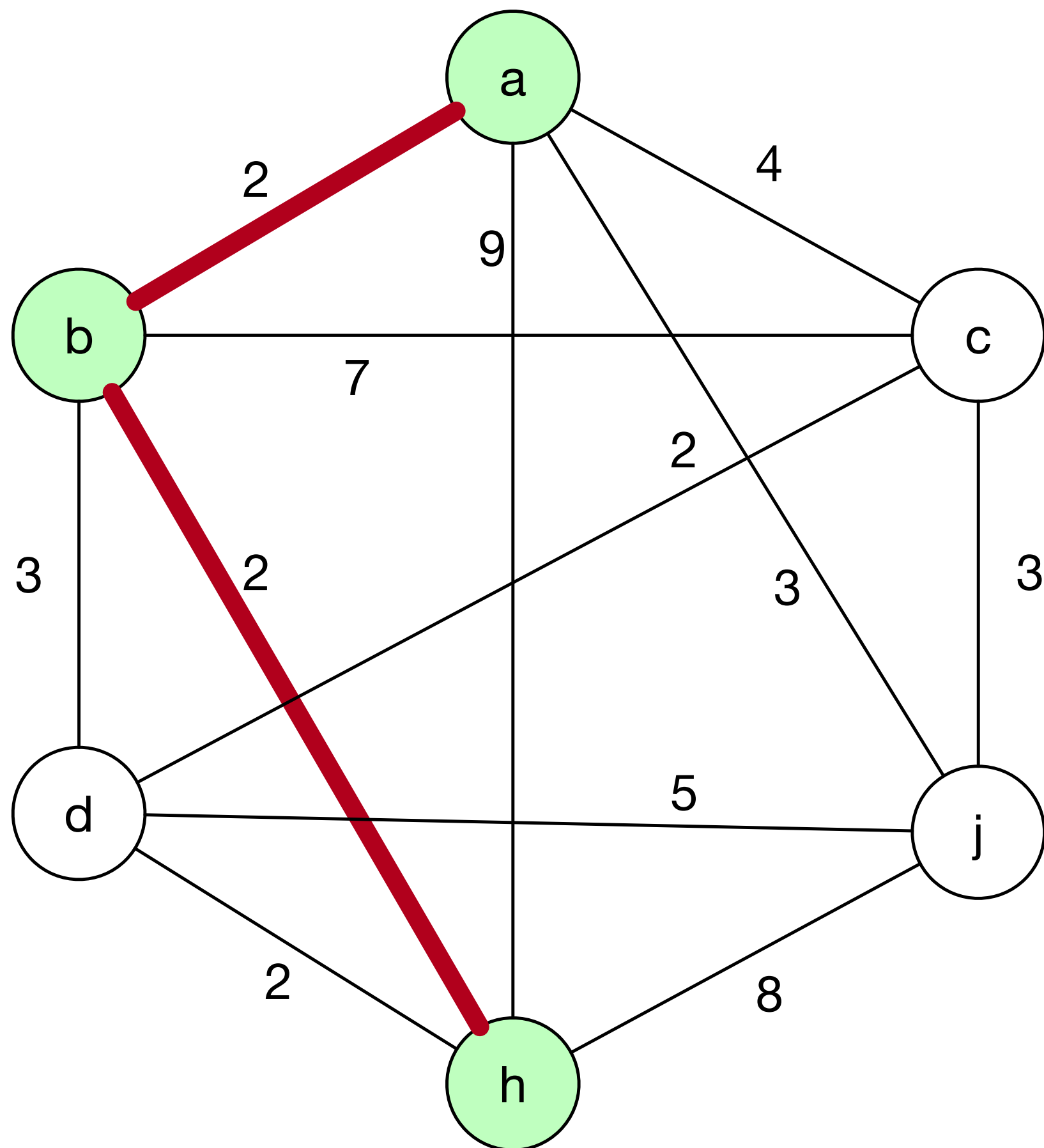
	lowest in V'	weight
a	-	-
b	-	-
c	a	4
d	h	2
j	a	3
h	-	-

Minimum spanning tree

Prim's algorithm

$$V' = \{b, a, h\}$$

$$E' = \{ab, bh\}$$



	a	b	c	d	j	h
a	0	2	4	0	3	9
b		0	7	3	0	2
c			0	2	3	0
d				0	5	2
j					0	8
h						0

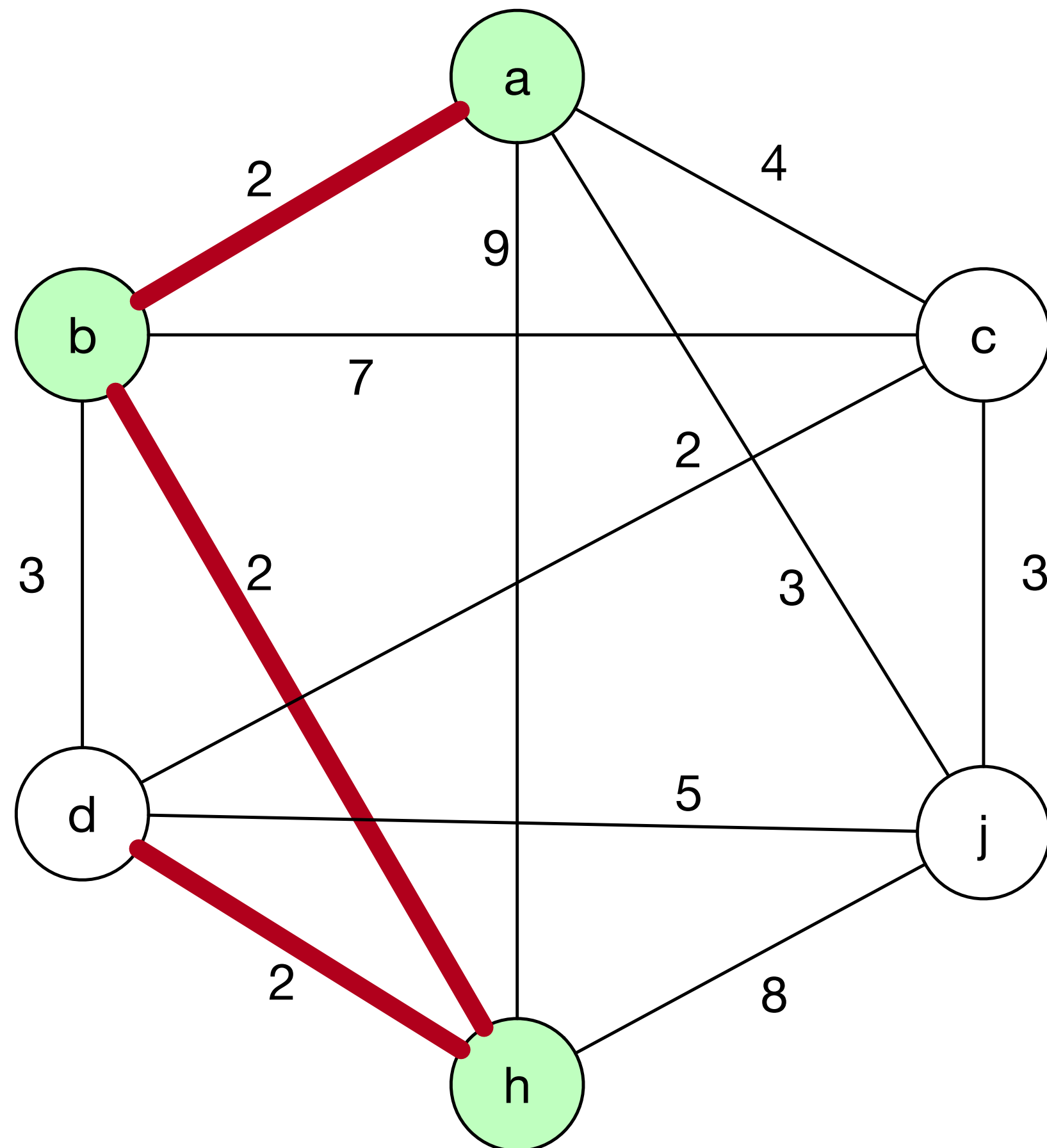
	lowest in V'	weight
a	-	-
b	-	-
c	a	4
d	h	2
j	a	3
h	-	-

Minimum spanning tree

Prim's algorithm

$$V' = \{b, a, h\}$$

$$E' = \{ab, bh, dh\}$$



	a	b	c	d	j	h
a	0	2	4	0	3	9
b		0	7	3	0	2
c			0	2	3	0
d				0	5	2
j					0	8
h						0

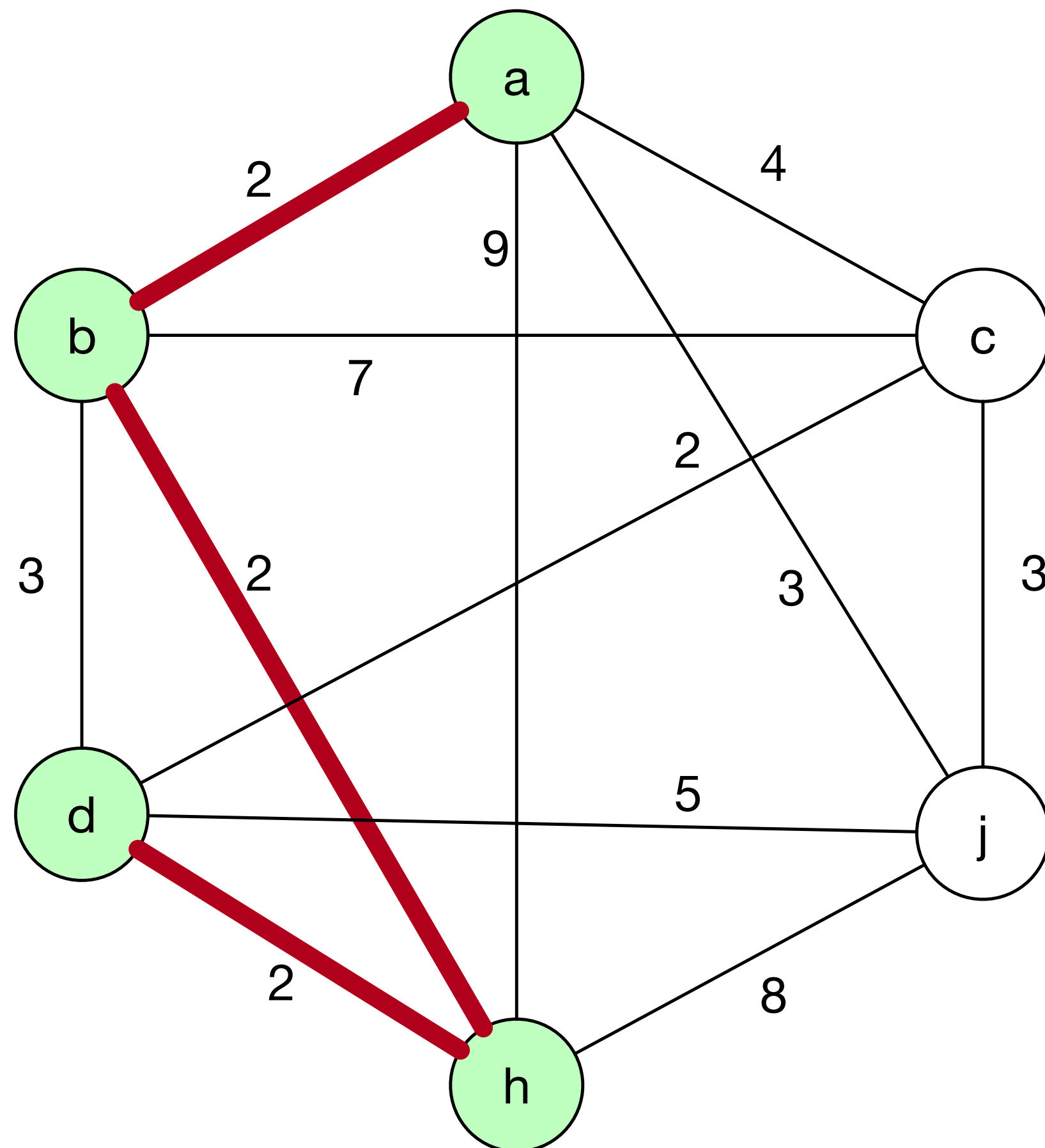
	lowest in V'	weight
a	-	-
b	-	-
c	a	4
d	h	2
j	a	3
h	-	-

Minimum spanning tree

Prim's algorithm

$$V' = \{b, a, h, d\}$$

$$E' = \{ab, bh, dh\}$$



	a	b	c	d	j	h
a	0	2	4	0	3	9
b		0	7	3	0	2
c			0	2	3	0
d				0	5	2
j					0	8
h						0

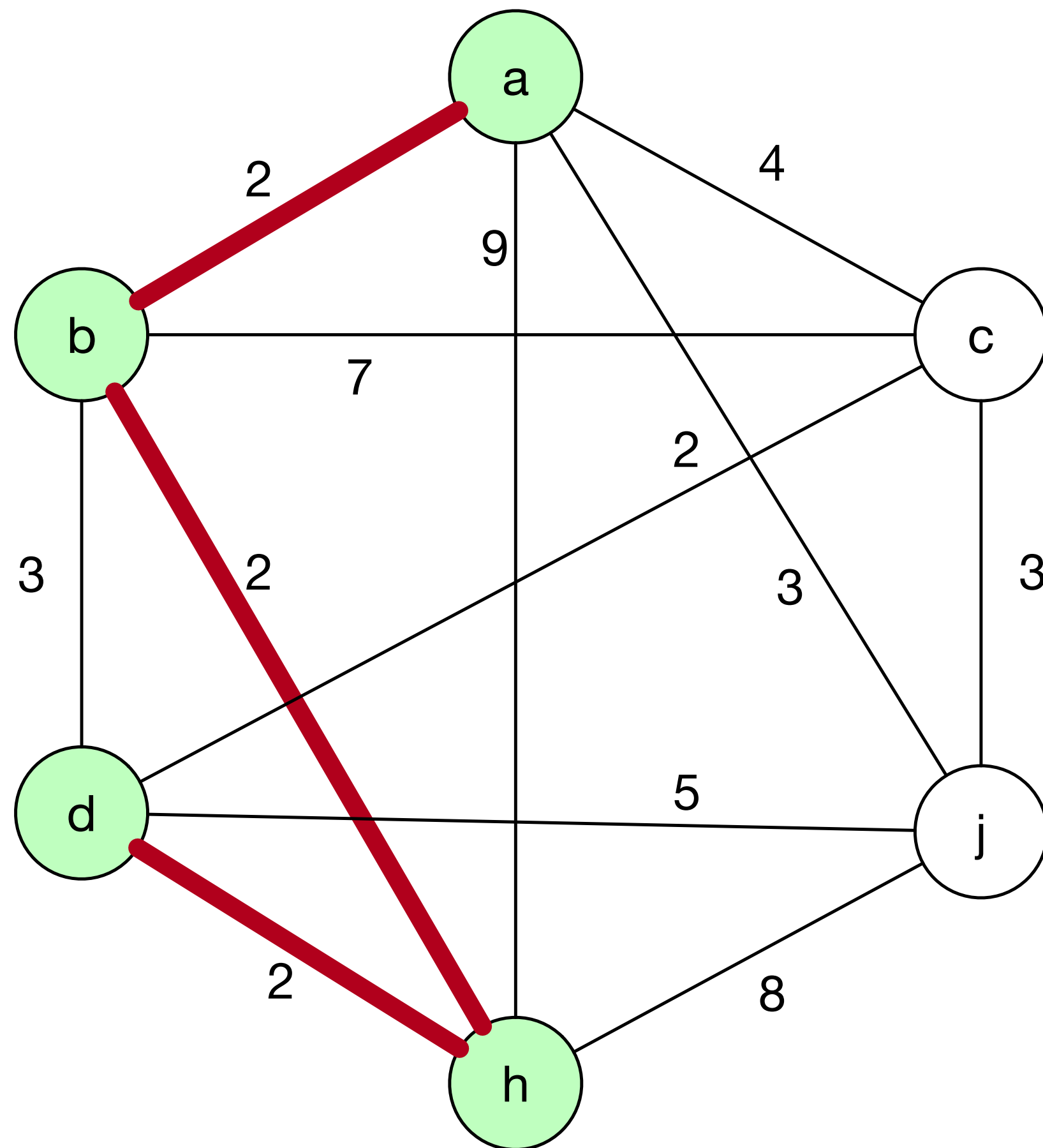
	lowest in V'	weight
a	-	-
b	-	-
c	a	4
d	h	2
j	a	3
h	-	-

Minimum spanning tree

Prim's algorithm

$$V' = \{b, a, h, d\}$$

$$E' = \{ab, bh, dh\}$$



	a	b	c	d	j	h
a	0	2	4	0	3	9
b		0	7	3	0	2
c			0	2	3	0
d				0	5	2
j					0	8
h						0

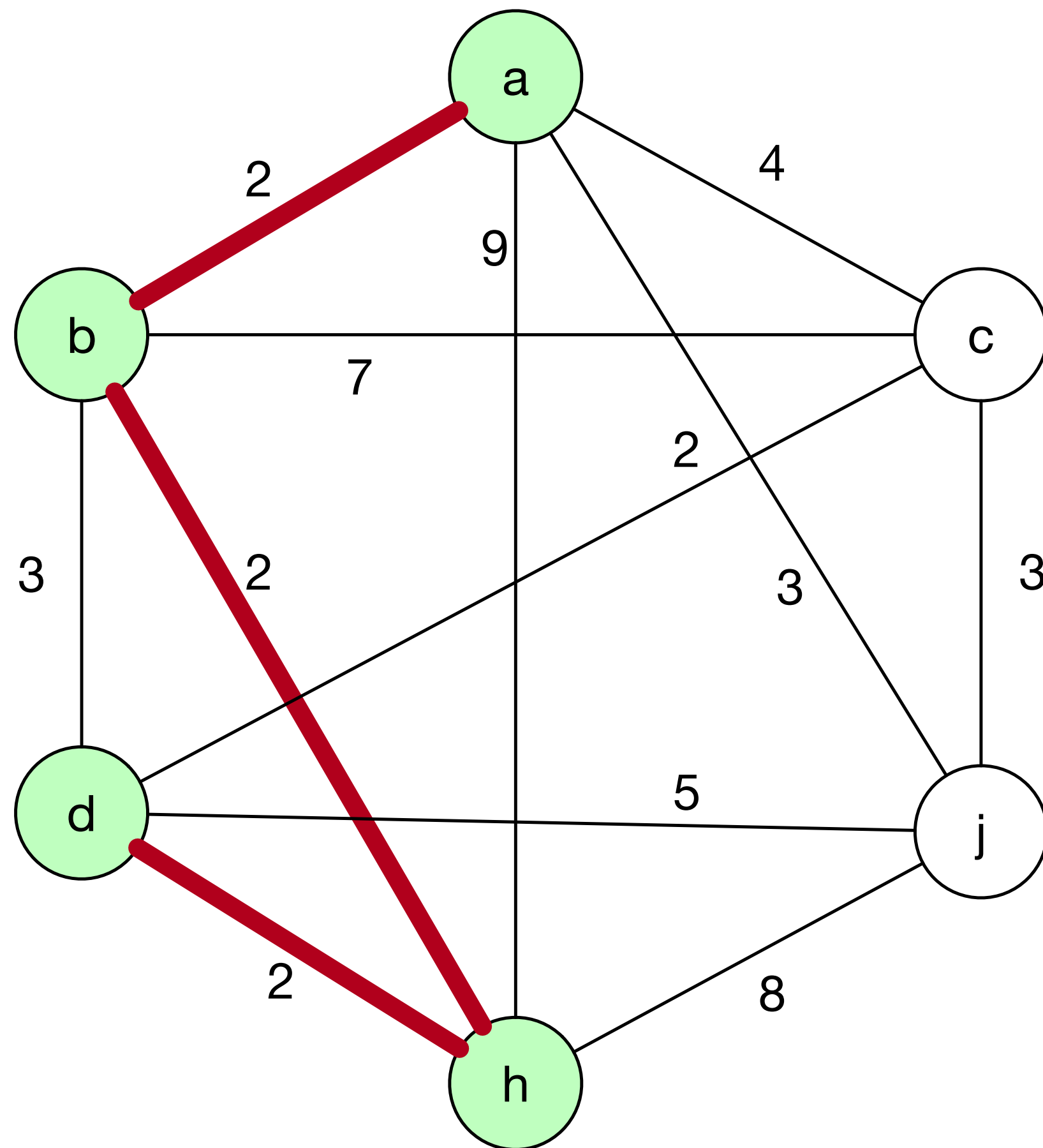
	lowest in V'	weight
a	-	-
b	-	-
c	d	2
d	h	2
j	a	3
h	-	-

Minimum spanning tree

Prim's algorithm

$$V' = \{b, a, h, d\}$$

$$E' = \{ab, bh, dh\}$$



	a	b	c	d	j	h
a	0	2	4	0	3	9
b		0	7	3	0	2
c			0	2	3	0
d				0	5	2
j					0	8
h						0

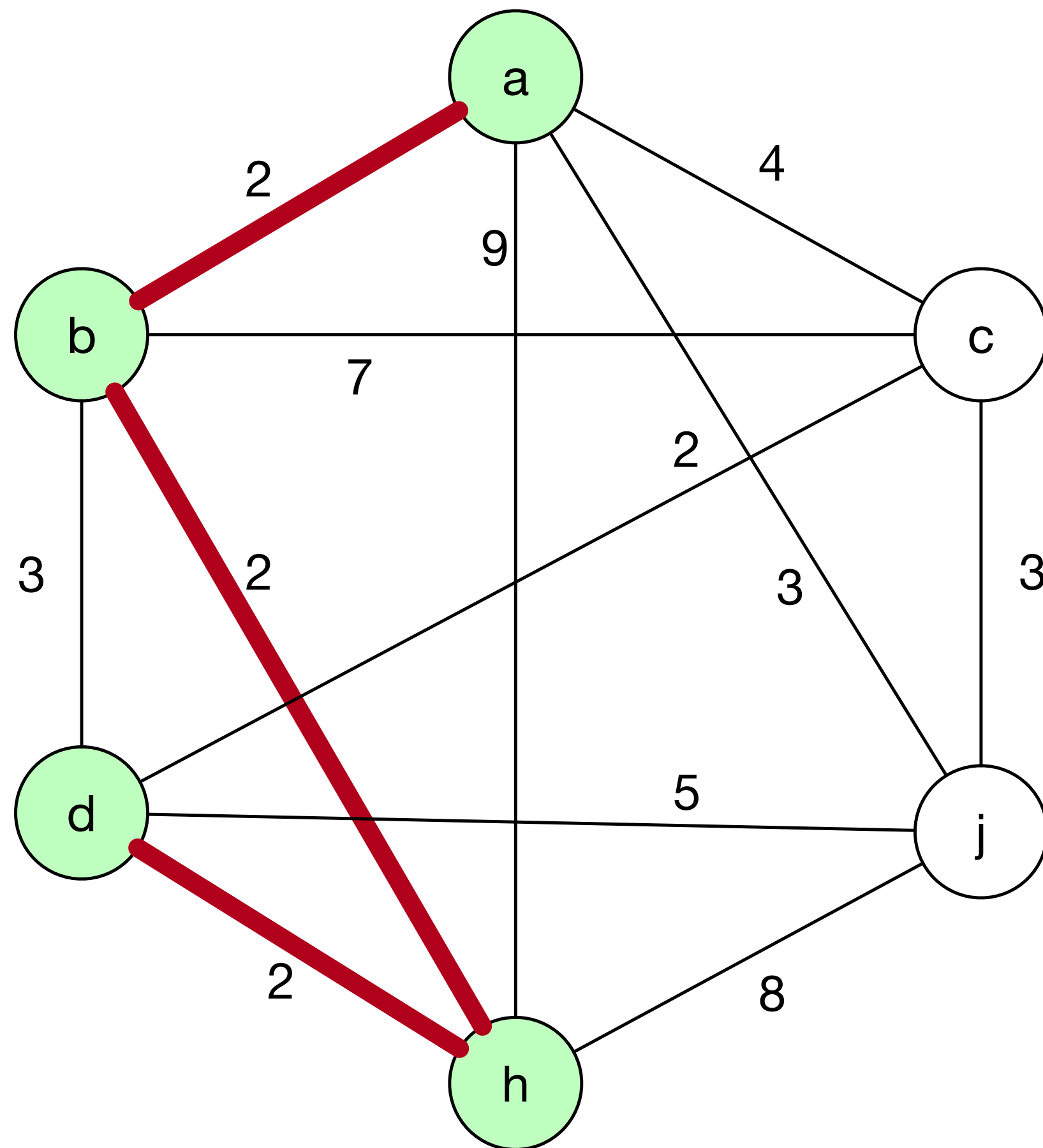
	lowest in V'	weight
a	-	-
b	-	-
c	d	2
d	-	-
j	a	3
h	-	-

Minimum spanning tree

Prim's algorithm

$$V' = \{b, a, h, d\}$$

$$E' = \{ab, bh, dh\}$$



	a	b	c	d	j	h
a	0	2	4	0	3	9
b		0	7	3	0	2
c			0	2	3	0
d				0	5	2
j					0	8
h						0

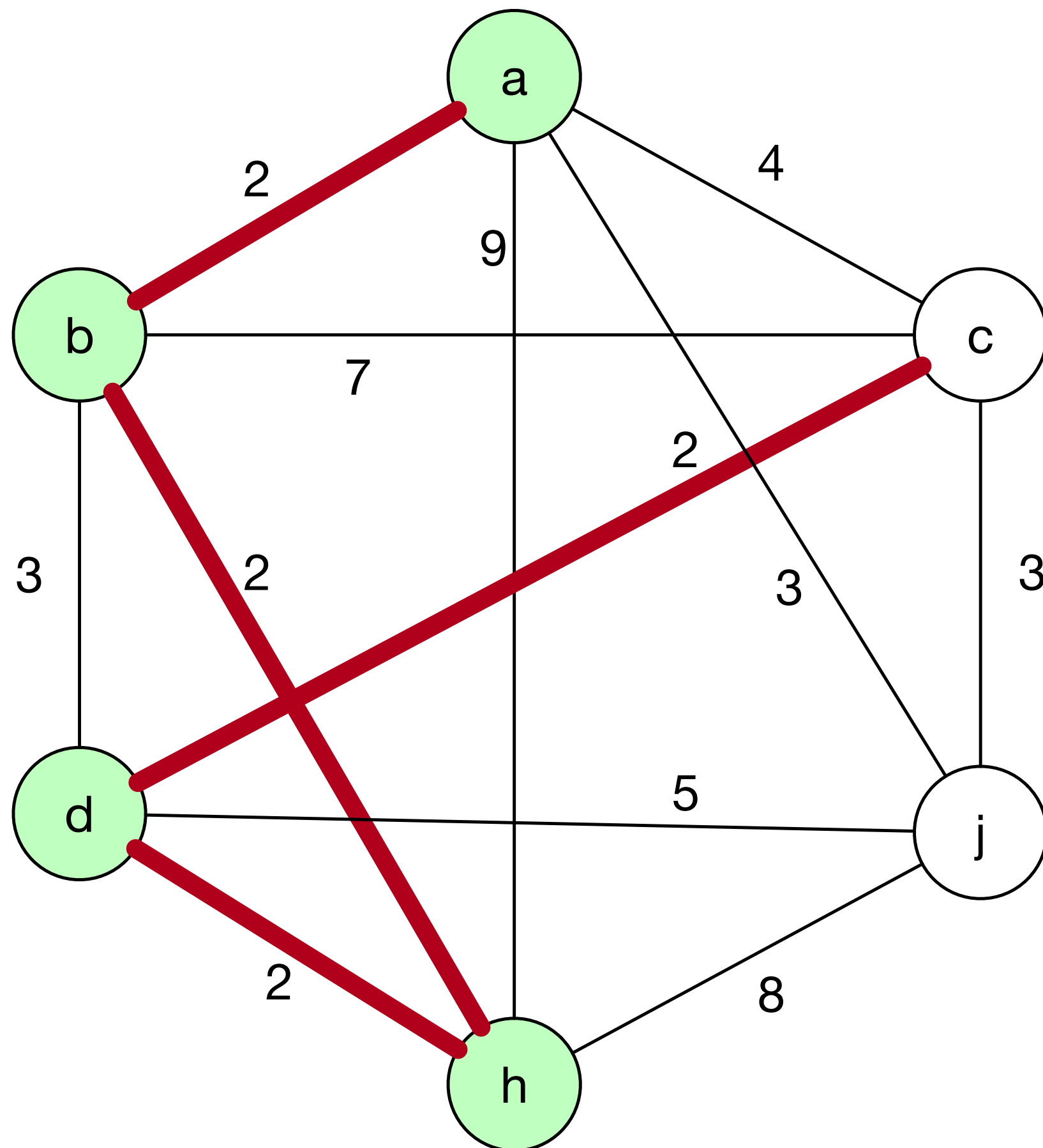
	lowest in V'	weight
a	-	-
b	-	-
c	d	2
d	-	-
j	a	3
h	-	-

Minimum spanning tree

Prim's algorithm

$$V' = \{b, a, h, d\}$$

$$E' = \{ab, bh, dh, cd\}$$



	a	b	c	d	j	h
a	0	2	4	0	3	9
b		0	7	3	0	2
c			0	2	3	0
d				0	5	2
j					0	8
h						0

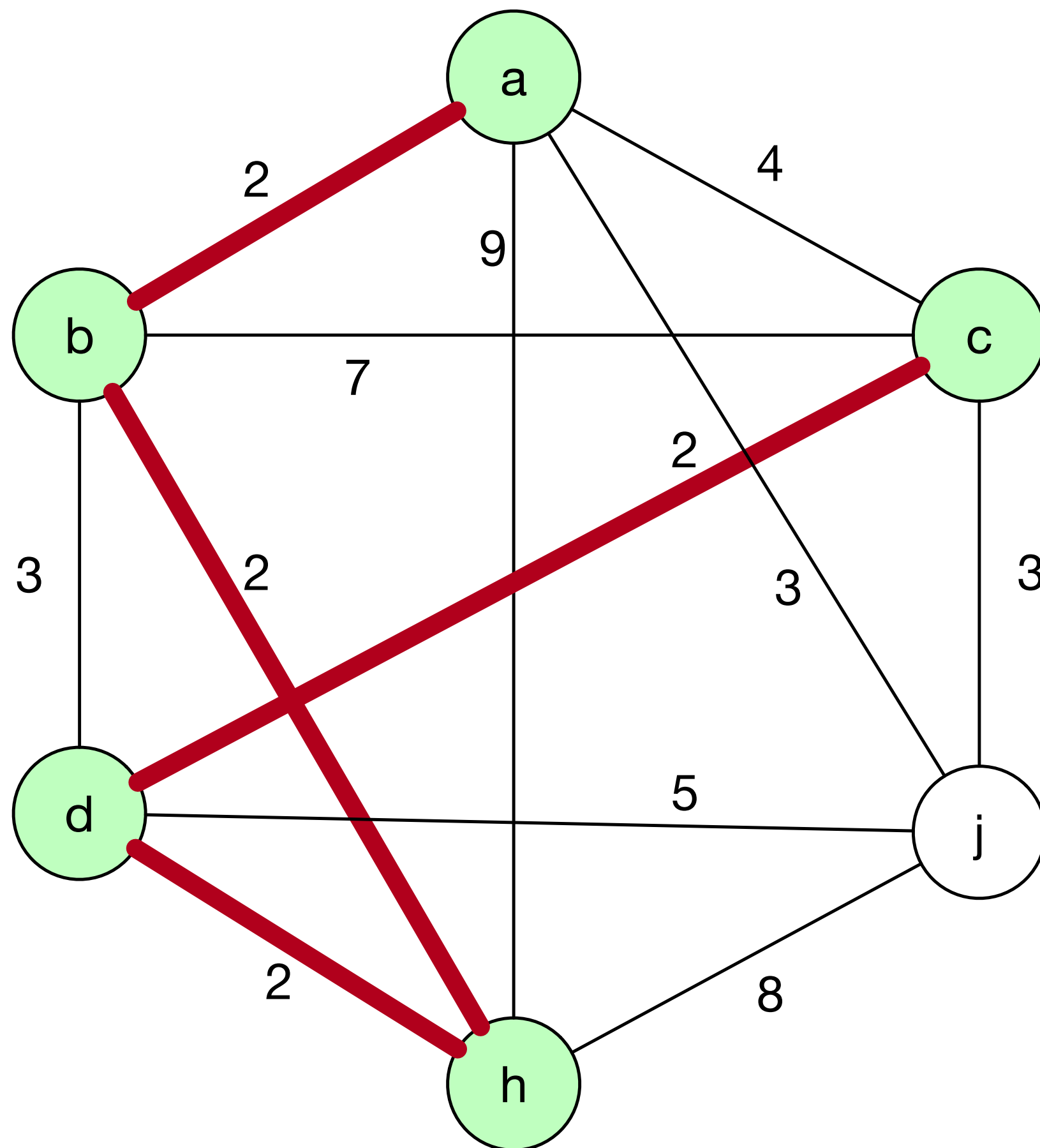
	lowest in V'	weight
a	-	-
b	-	-
c	d	2
d	-	-
j	a	3
h	-	-

Minimum spanning tree

Prim's algorithm

$$V' = \{b, a, h, d, c\}$$

$$E' = \{ab, bh, dh, cd\}$$



	a	b	c	d	j	h
a	0	2	4	0	3	9
b		0	7	3	0	2
c			0	2	3	0
d				0	5	2
j					0	8
h						0

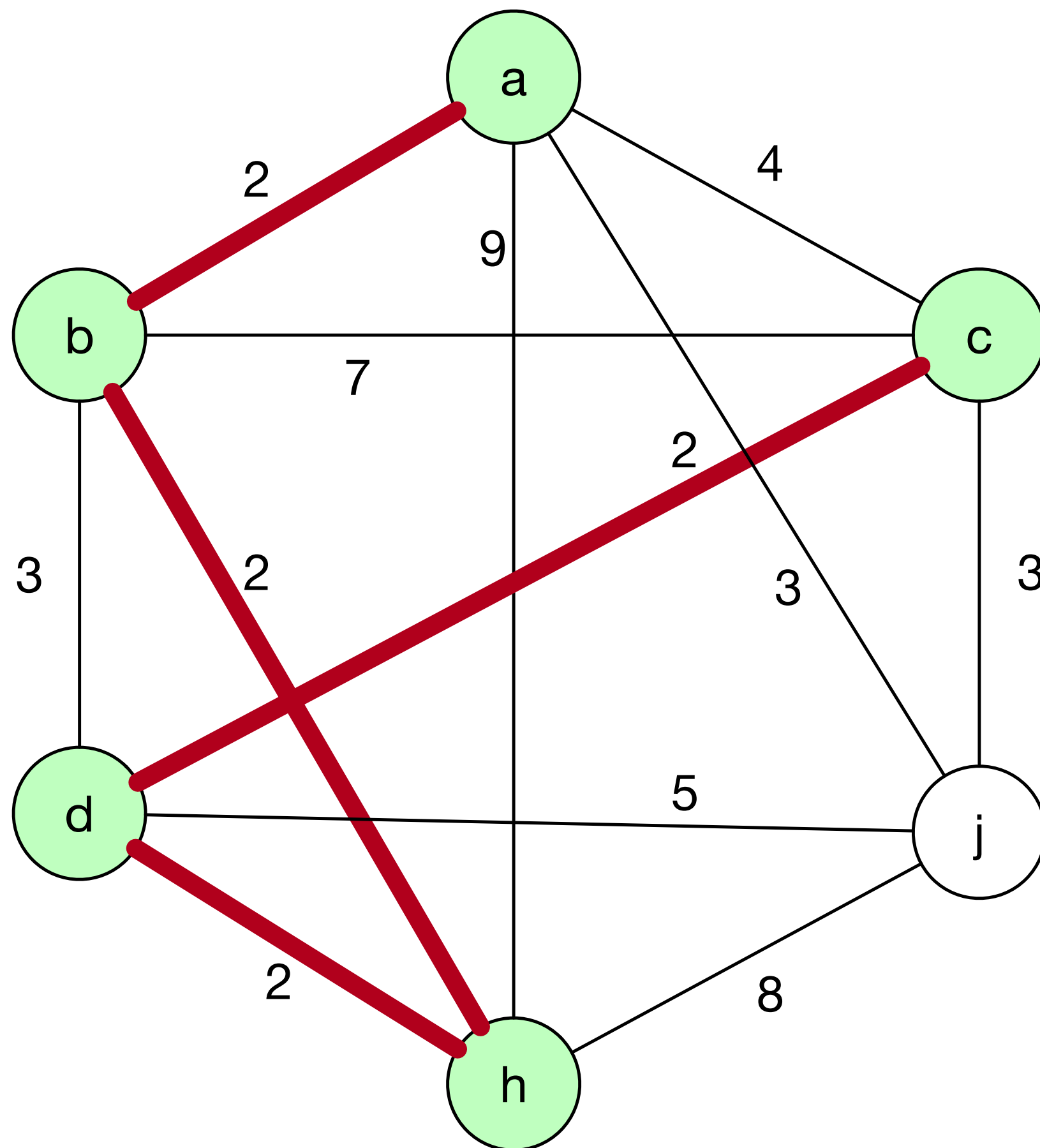
	lowest in V'	weight
a	-	-
b	-	-
c	d	2
d	-	-
j	a	3
h	-	-

Minimum spanning tree

Prim's algorithm

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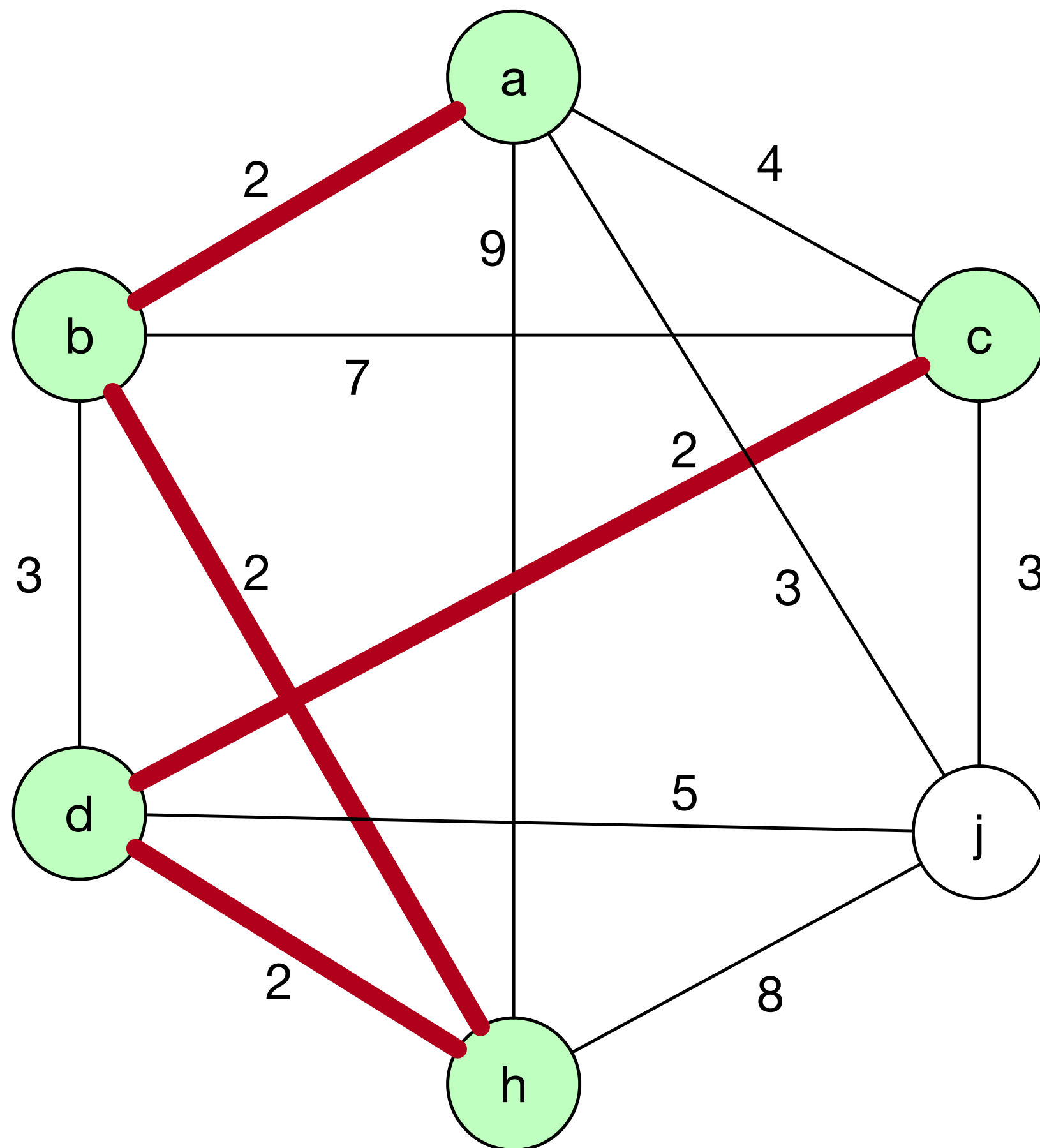
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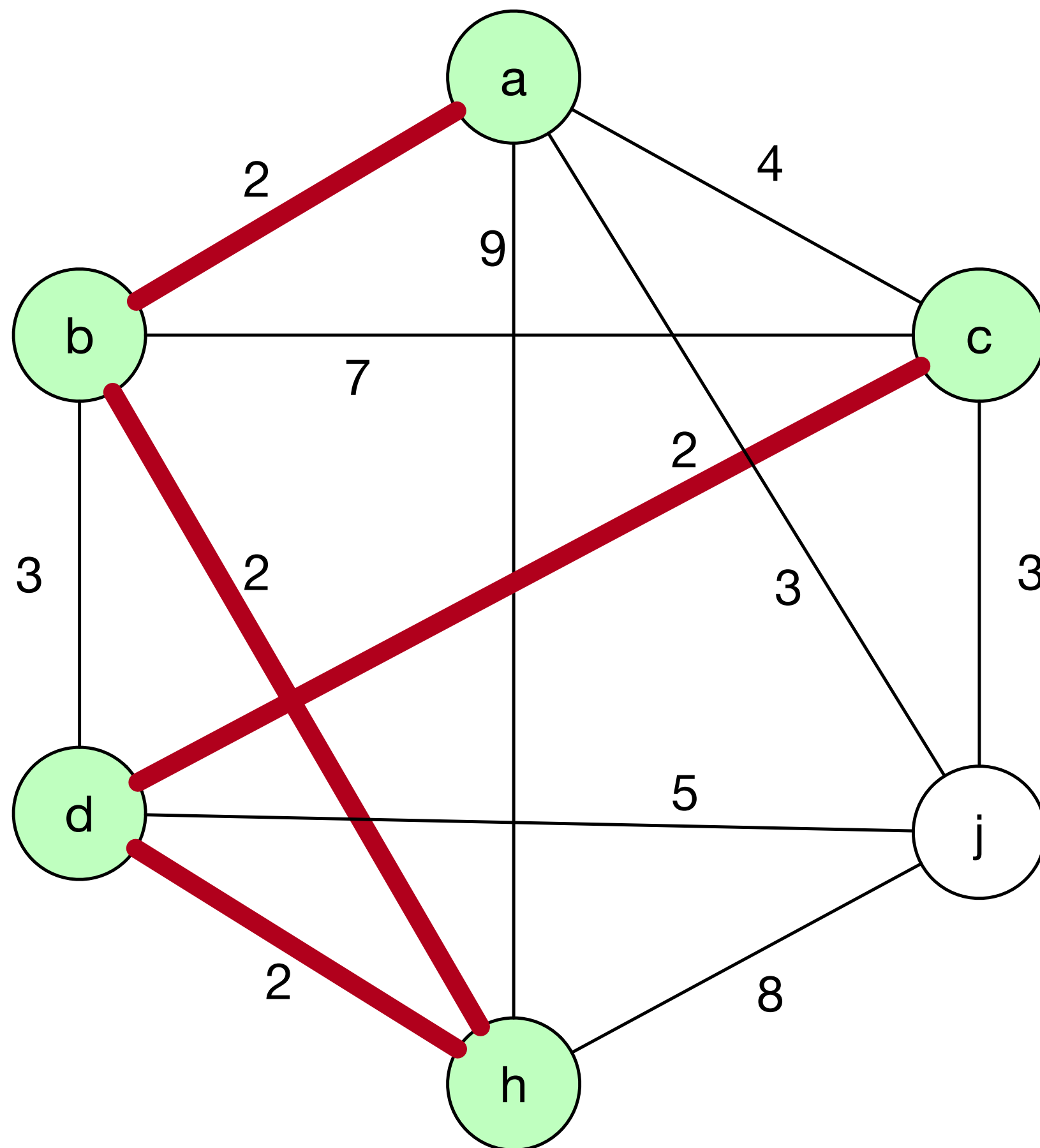
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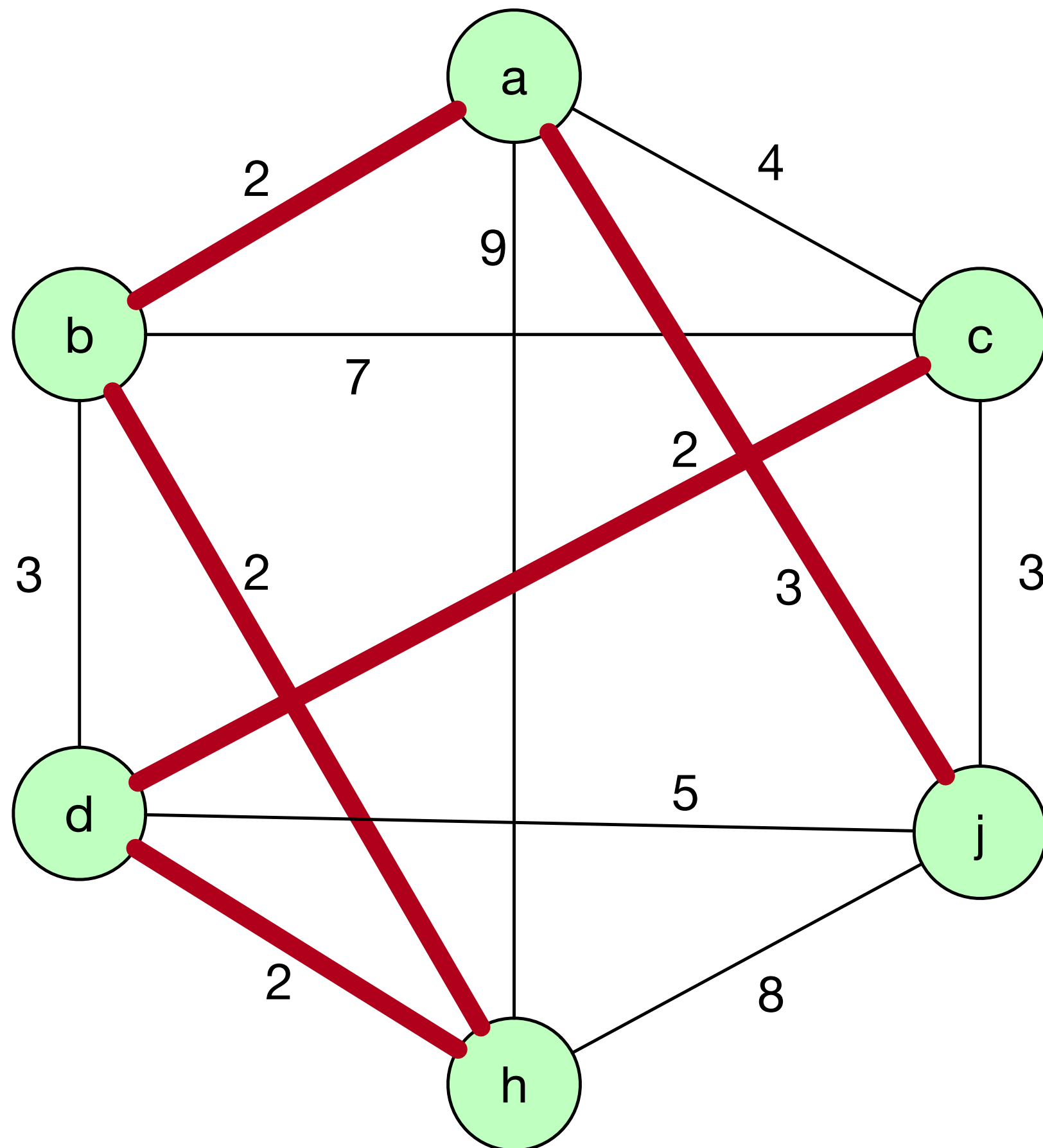
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Adjacency matrix is best if the graph is dense (high ratio of edges to nodes).

Min heap is best if graph is sparse (low ratio of edges to nodes).