PoCS What's The Story?

Principles of Complex Systems, CSYS/MATH 300
University of Vermont, Fall 2014
Assignment 8 • code name: Octarine

Dispersed: Friday, October 31, 2014.
Due: By start of lecture, 1:00 pm, Thursday, November 6, 2014.
Some useful reminders:
Instructor: Peter Dodds
Office: Farrell Hall, second floor, Trinity Campus
E-mail: peter.dodds@uvm.edu
Office hours: 2:30 pm to 3:45 pm on Tuesday, 12:30 pm to 2:00 pm on Wednesday
Course website: http://www.uvm.edu/∼pdodds/teaching/courses/2014-08UVM-300

All parts are worth 3 points unless marked otherwise. Please show all your workingses clearly and list the names of others with whom you collaborated.

Graduate students are requested to use LaTeX (or related TeX variant).

Also: Please submit your project's current state of development.

1. (3 + 3 + 3 + 3 + 3) *More on the power law stuff:*

   We take a look at the 80/20 rule, 1 per centers, and similar concepts.

   Take x to be the wealth held by an individual in a population of n people, and the number of individuals with wealth between x and $x + \mathrm{d}x$ to be approximately $N(x)\mathrm{d}x$.

   Given a power-law size frequency distribution $N(x) = cx^{-\gamma}$ where $x_{\min} \ll x \ll \infty$, determine the value of γ for which the so-called 80/20 rule holds.

   In other words, find γ for which the bottom 4/5 of the population holds 1/5 of the overall wealth, and the top 1/5 holds the remaining 4/5.

   Assume the mean is finite, i.e., $\gamma > 2$.

   (a) Determine the total wealth W in the system given $\int_{x_{\min}}^{\infty} \mathrm{d}x N(x) = n$.

   (b) Imagine that $100q$ percent of the population holds $100(1 - r)$ percent of the wealth.

   Show γ depends on q and r as

$$\gamma = 1 + \frac{\ln \frac{1}{(1-q)}}{\ln \frac{1}{(1-q)} - \ln \frac{1}{r}}.$$

(c) Given the above, is every pairing of q and r possible?

(d) Find γ for the 80/20 requirement ($q = r = 4/5$).

(e) For the "80/20" γ you find, determine how much wealth $100q$ percent of the population possesses as a function of q and plot the result.

2. (3 + 3 + 3) Optional:

   Solve Krapivsky-Redner's model for the pure linear attachment kernel $A_k = k$.

   Starting point:

$$n_k = \frac{1}{2}(k-1)n_{k-1} - \frac{1}{2}kn_k + \delta_{k1}$$

   with $n_0 = 0$.

   (a) Determine n_1.

   (b) Find a recursion relation for n_k in terms of n_{k-1}.

   (c) Now find

$$n_k = \frac{4}{k(k+1)(k+2)}$$

   for all k and hence determine γ.

3. (3 + 3) Optional:

   From lectures:

   (a) Starting from the recursion relation

$$n_k = \frac{A_{k-1}}{\mu + A_k} n_{k-1},$$

   and $n_1 = \mu/(\mu + A_1)$, show that the expression for n_k for the Krapivsky-Redner model with an asymptotically linear attachment kernel A_k is:

$$\frac{\mu}{A_k} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}.$$

   (b) Now show that if $A_k \to k$ for $k \to \infty$ (or for large k), we obtain $n_k \to k^{-\mu-1}$.

4. (3 + 3 + 3) Optional:

   From lectures, complete the analysis for the Krapivsky-Redner model with attachment kernel:

$$A_1 = \alpha \text{ and } A_k = k \text{ for } k \geq 2.$$

Find the scaling exponent $\gamma = \mu + 1$ by finding μ. From lectures, we assumed a linear growth in the sum of the attachment kernel weights $\mu t = \sum_{k=1}^{\infty} N_k(t) A_k$, with $\mu = 2$ for the standard kernel $A_k = k$.

We arrived at this expression for μ which you can use as your starting point:

$$1 = \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

(a) Show that the above expression leads to

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

Hint: you'll want to separate out the $j = 1$ case for which $A_j = \alpha$.

(b) Now use result that [1]

$$\sum_{k=2}^{\infty} \frac{\Gamma(a+k)}{\Gamma(b+k)} = \frac{\Gamma(a+2)}{(b-a-1)\Gamma(b+1)}$$

to find the connection

$$\mu(\mu - 1) = 2\alpha,$$

and show this leads to

$$\mu = \frac{1 + \sqrt{1 + 8\alpha}}{2}.$$

(c) Interpret how varying α affects the exponent γ, explaining why $\alpha < 1$ and $\alpha > 1$ lead to the particular values of γ that they do.

References

[1] P. L. Krapivsky and S. Redner. Organization of growing random networks. *Phys. Rev. E*, 63:066123, 2001. pdf