PoCS — What's The Story?

Principles of Complex Systems, CSYS/MATH 300
University of Vermont, Fall 2014
Assignment 4 • code name: Adora Belle

Dispersed: Thursday, September 18, 2014.
Due: By start of lecture, 1:00 pm, ~~Thursday, September 25~~ Tuesday, September 30, 2014.
Some useful reminders:
Instructor: Peter Dodds
Office: Farrell Hall, second floor, Trinity Campus
E-mail: peter.dodds@uvm.edu
Office hours: 2:30 pm to 3:45 pm on Tuesday, 12:30 pm to 2:00 pm on Wednesday
Course website: http://www.uvm.edu/∼pdodds/teaching/courses/2014-08UVM-300

All parts are worth 3 points unless marked otherwise. Please show all your working clearly and list the names of others with whom you collaborated.

Graduate students are requested to use LaTeX (or related TeX variant).

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1. (3 + 3 + 3)

   A courageous coding festival:

   Code up the discrete HOT model in 1-d. Let's see if we find any of these super-duper power laws everyone keeps talking about. We'll follow the same approach as the 2-d forest discussed in lectures.

   Main goal: extract yield curves as a function of the design D parameter as described below.

   Suggested simulations elements:

   - $N = 10^4$ as a start. Then see if $N = 10^5$ or $N = 10^6$ is possible.
   - Start with no trees.
   - Probability of a spark at the ith site: $P(i) \propto e^{-i/\ell}$ where i is tree position ($i = 1$ to N). (You will need to normalize this properly.) The quantity ℓ is the characteristic scale for this distribution; try $\ell = 2 \times 10^5$.
   - Consider a design problem of $D = 1$, 2, $N^{1/2}$, and N. (If $N^{1/2}$ and N are too much, you can drop them. Perhaps sneak out to $D = 3$.) Recall that the design problem is to test D randomly chosen placements of the next tree against the spark distribution.
   - For each test tree, measure the average yield (number of trees left) with $n = 100$ randomly selected sparks. Select the tree location with the highest average yield and plant a tree there.

- Add trees until the linear forest is full, measuring average yield as a function of trees added.
- Only trees and adjacent trees burn. In effect, you will be burning un-treed intervals of the line (much less complicated than 2-d).

(a) Plot the yield curves for each value of D.

(b) Identify peak yield for each value of D.

(c) Plot distributions of connected tree interval sizes at peak yield (you will have to rebuild forests and stop at the peak yield value of D to find these distributions.

Hint: keeping a list of un-treed locations will make choosing the next location easier. Hopefully.

Highly optional territory: Code up the 2-d version and share the goodness on Github.

2. The discrete version of HOT theory:

   From lectures, we had the follwoing.

   Cost: Expected size of 'fire' in a d-dimensional lattice:

$$C_{\textrm{fire}} \propto \sum_{i=1}^{N_{\textrm{sites}}} (p_i a_i) a_i = \sum_{i=1}^{N_{\textrm{sites}}} p_i a_i^2,$$

   where a_i = area of ith site's region, and p_i = avg. prob. of fire at site in ith site's region.

   From lectures, the constraint for building and maintaining $(d-1)$-dimensional firewalls in d-dimensions is

$$C_{\textrm{firewalls}} \propto \sum_{i=1}^{N_{\textrm{sites}}} a_i^{(d-1)/d} a_i^{-1},$$

   where we are assuming isometry.

   Using Lagrange Multipliers, safety goggles, rubber gloves, a pair of tongs, and a maniacal laugh, determine that:

$$p_i \propto a_i^{-\gamma} = a_i^{-(2+1/d)}.$$

3. (3 + 3 + 3) *Highly Optimized Tolerance:*

   This question is based on Carlson and Doyle's 1999 paper "Highly optimized tolerance: A mechanism for power laws in design systems" [1]. In class, we made our way through a discrete version of a toy HOT model of forest fires. This paper revolves around the equivalent continuous model's derivation. You do not have to perform the derivation but rather carry out some manipulations of probability distributions using their main formula.

Our interest is in Table I on p. 1415:

$p(x)$	$p_{\text{cum}}(x)$	$P_{\text{cum}}(A)$
$x^{-(q+1)}$	x^{-q}	$A^{-\gamma(1-1/q)}$
e^{-x}	e^{-x}	$A^{-\gamma}$
e^{-x^2}	$x^{-1}e^{-x^2}$	$A^{-\gamma}[\log(A)]^{-1/2}$

and Equation 8 on the same page:

$$P_{\geq}(A) = \int_{p^{-1}(A^{-\gamma})}^{\infty} p(\mathbf{x})\mathrm{d}\mathbf{x} = p_{\geq}\left(p^{-1}\left(A^{-\gamma}\right)\right),$$

where $\gamma = \alpha + 1/\beta$ and we'll write $P_{\geq}$ for P_{cum}.

Please note that $P_{\geq}(A)$ for $x^{-(q+1)}$ is not correct. Find the right one!

Here, $A(\mathbf{x})$ is the area connected to the point $\mathbf{x}$ (think connected patch of trees for forest fires). The cost of a 'failure' (e.g., lightning) beginning at $\mathbf{x}$ scales as $A(\mathbf{x})^{\alpha}$ which in turn occurs with probability $p(\mathbf{x})$. The function p^{-1} is the inverse function of p.

Resources associated with point $\mathbf{x}$ are denoted as $R(\mathbf{x})$ and area is assumed to scale with resource as $A(\mathbf{x}) \sim R^{-\beta}(\mathbf{x})$.

Finally, $p_{\geq}$ is the complementary cumulative distribution function for p.

As per the table, determine $p_{\geq}(x)$ and $P_{\geq}(A)$ for the following (3 pts each):

(a) $p(x) = cx^{-(q+1)}$,

(b) $p(x) = ce^{-x}$, and

(c) $p(x) = ce^{-x^2}$.

Note that these forms are for the tails of p only, and you should incorporate a constant of proportionality c, which is not shown in the paper.

4. In lectures on lognormals and other heavy-tailed distributions, we came across a super fun and interesting integral when considering organization size distributions arising from growth processes with variable lifespans.

   Show that

$$P(x) = \int_{t=0}^{\infty} \lambda e^{-\lambda t} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln \frac{x}{m})^2}{2t}\right) \mathrm{d}t$$

leads to:

$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln \frac{x}{m})^2}},$$

and therefore, surprisingly, two different scaling regimes. Enjoyable suffering may be involved. Really enjoyable suffering. But many monks have found a way so you should follow their path laid out below.

Hints and steps:

- Make the substitution $t = u^2$ to find an integral of the form (excluding a constant of proportionality)

  $$I_1(a, b) = \int_0^\infty \exp\left(-au^2 - b/u^2\right) \mathrm{d}u$$

  where in our case $a = \lambda$ and $b = (\ln \frac{x}{m})^2/2$.
- Substitute $au^2 = t^2$ into the above to find

  $$I_1(a, b) = \frac{1}{\sqrt{a}} \int_0^\infty \exp\left(-t^2 - ab/t^2\right) \mathrm{d}t$$

- Now work on this integral:

  $$I_2(r) = \int_0^\infty \exp\left(-t^2 - r/t^2\right) \mathrm{d}t$$

  where $r = ab$.
- Differentiate I_2 with respect to r to create a simple differential equation for I_2 You will need to use the substitution $u = \sqrt{r}/t$ and your differential equation should be of the (very simple) form

  $$\frac{\mathrm{d}I_2(r)}{\mathrm{d}r} = -(\text{something})I_2(r).$$

- Solve the differential equation you find. To find the constant of integration, you can evaluate $I_2(0)$ separately:

  $$I_2(0) = \int_0^\infty \exp(-t^2)\mathrm{d}t,$$

  where our friend $\Gamma(1/2)$ comes into play.

References

[1] J. M. Carlson and J. Doyle. Highly optimized tolerance: A mechanism for power laws in designed systems. *Phys. Rev. E*, 60(2):1412–1427, 1999. pdf